

Transgressive loop group extensions

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Abstract

A central extension of the loop group of a Lie group is called transgressive, if it corresponds under transgression to a degree four class in the cohomology of the classifying space of the Lie group. Transgressive loop group extensions are those that can be explored by finite-dimensional, higher-categorical geometry over the Lie group. We show how transgressive central extensions can be characterized in a loop-group theoretical way, in terms of loop fusion and thin homotopy equivariance.

Contents

1	Introduction	1
2	Fusion and thin homotopy equivariance over loop groups	3
3	Features of fusion and thin homotopy equivariance	10
3.1	Flat loops and retraction	10
3.2	Loop concatenation	11
3.3	Disjoint-commutativity	12
4	Integrable thin homotopy equivariant structures	13
5	Transgression-regression machine	18
5.1	Multiplicative bundle gerbes	18
5.2	Transgressive central extensions	21
5.3	Regression and equivalence result	24
6	Segal-Witten reciprocity	27
A	Regression of trivial fusion bundles	30
	Table of terminology	35
	References	36

1 Introduction

The present article is about a Lie group G , its loop group $LG := C^\infty(S^1, G)$ and central extensions

$$1 \longrightarrow \mathrm{U}(1) \longrightarrow \mathcal{L} \longrightarrow LG \longrightarrow 1$$

in the category of Fréchet Lie groups. Some central extensions \mathcal{L} can be obtained from structure over G called *multiplicative bundle gerbe with connection* via a procedure called *transgression*. These central

extensions are called *transgressive*. In the case of G compact, transgression induces a map

$$H^4(BG, \mathbb{Z}) \longrightarrow \left\{ \begin{array}{l} \text{Isomorphism classes of} \\ \text{central extensions of } LG \end{array} \right\},$$

and a central extension is transgressive if and only if it is in the image of that map. An example of a transgressive central extension is the universal central extension of the loop group of a compact simply-connected Lie group G : it corresponds to a generator of $H^4(BG, \mathbb{Z}) \cong \mathbb{Z}$.

The goal of this article is to characterize transgressive central extensions for arbitrary connected Lie groups in purely loop group-theoretic terms. For this purpose we consider two relations on LG :

1. thin homotopy: a homotopy between two loops is called thin, if its differential has nowhere full rank; these are homotopies that sweep out a surface of zero area.
2. loop fusion: it relates two loops that share a common line segment to a new loop with that segment deleted.

We introduce the notion of a *thin fusion extension*: a central extension

$$1 \longrightarrow \mathrm{U}(1) \longrightarrow \mathcal{L} \longrightarrow LG \longrightarrow 1$$

in which both relations are lifted in a consistent way to \mathcal{L} . Thin fusion extensions form a subclass of central extensions of LG with several interesting properties. For example, we show that they are disjoint-commutative: if $p_1, p_2 \in \mathcal{L}$ project to loops supported on disjoint subintervals of S^1 , then they commute, $p_1 \cdot p_2 = p_2 \cdot p_1$.

The main results of this paper are summarized in the following main theorem.

Theorem A. *Let G be a connected Lie group. A central extension \mathcal{L} of LG is transgressive if and only if it can be equipped with the structure of a thin fusion extension. Moreover, transgression is a group isomorphism*

$$\left\{ \begin{array}{l} \text{Isomorphism classes of} \\ \text{multiplicative bundle gerbes} \\ \text{over } G \text{ that admit connections} \end{array} \right\} \cong \left\{ \begin{array}{l} \text{Isomorphism classes} \\ \text{of thin fusion} \\ \text{extensions of } LG \end{array} \right\}.$$

If G is compact, both groups are isomorphic to $H^4(BG, \mathbb{Z})$.

As a consequence of Theorem A, we obtain that transgressive central extensions are disjoint-commutative; this provides an accessible necessary condition for the transgressivity of a central extension.

The present work is a contribution to the programme of exploring the geometry of loop groups via finite-dimensional, higher geometry over Lie groups. Our main theorem determines the class of central extensions of loop groups that are accessible by such methods: thin fusion extensions.

Transgression of gerbes has first been defined by Gawędzki in relation with two-dimensional conformal field theories [Gaw88], and then by Brylinski and McLaughlin in the setting of sheaves of groupoids [Bry93, BM94]. The multiplicative bundle gerbes we use here have been introduced by Carey et al. in [CJM⁺05], and transgression of those has been developed in [Wal10].

The question which central extensions are transgressive has been studied before by Brylinski and McLaughlin [BM94, BM96]. For connected semisimple complex Lie groups, they obtained a characterization in terms of the so-called Segal-Witten reciprocity law. In [BM94] it is incorrectly stated [Bry]

that this reciprocity law also holds for non-complex Lie groups. Indeed, we provide a counterexample to that statement and prove that only a *weaker* version of the reciprocity law holds for transgressive central extensions of general Lie groups. We also provide an example of a central extension that is not transgressive and yet satisfies this weaker version of the reciprocity law. We come to the conclusion that no version of the reciprocity law appropriately characterizes transgressive central extensions of general Lie groups. It was the main motivation for writing this article to attack the open characterization problem from a different angle, namely via fusion and thin homotopy equivariance.

The results of this article are based on previous work on transgression [Wal12b, Walb, Wal12c]. A summary of these three papers on only three pages can be found in [Wal12c, Section 1]. The main result is that transgression for general smooth manifolds X establishes an equivalence between various categories of gerbes over X and corresponding categories of S^1 -bundles over the free loop space LX , equipped with structure rendering them compatible with fusion and thin homotopy. The present paper can be seen as an extension of these results to a multiplicative setting.

The organization of the present paper is as follows. In Section 2 we introduce the basic definitions of fusion and thin homotopy equivariance in loop group extensions. In Section 3 we provide a list of features that follow from the presence of these structures, among them disjoint-commutativity (Theorem 3.3.1). In Section 4 we formulate an integrability condition for thin homotopy equivariant structures on which our notion of thin fusion extensions is based (Definition 4.7). In Section 5 we discuss multiplicative bundle gerbes and their transgression, and introduce our definition of transgressive central extensions (Definition 5.2.1). Then we give a proof of Theorem A (split into Proposition 5.2.2 and Corollaries 5.3.2, 5.3.4). In Section 6 we prove our weaker version of the Segal-Witten reciprocity law (Theorem 6.3) and provide the two examples that indicate the above-mentioned problems (Examples 6.6 and 6.7).

Throughout the paper, we continuously look at two classes of examples: an explicit model of the universal central extension of the loop group of a compact simply-connected Lie group, and various central extensions of $LU(1)$, of which some turn out to be transgressive and others not. For the convenience of the reader we include on Page 35 a table summarizing some terminology we use in this paper.

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2 Fusion and thin homotopy equivariance over loop groups

In this section G is a Lie group and

$$1 \longrightarrow U(1) \longrightarrow \mathcal{L} \longrightarrow LG \longrightarrow 1$$

is a central extension of Fréchet Lie groups [PS86]. We introduce structures on \mathcal{L} that lift loop fusion and thin homotopy, and discuss the interplay between them.

By PG we denote the set of paths $\gamma : [0, 1] \longrightarrow G$ with sitting instants, i.e. they are constant near the endpoints. PG is not a Fréchet manifold, but can be treated as a *diffeological space*. Instead of charts, a diffeological space X has plots: maps $c : U \longrightarrow X$ defined on open subsets $U \subseteq \mathbb{R}^n$, for all $n \in \mathbb{N}$, satisfying three natural axioms, see e.g. [IZ13]. In case of PG , a map $c : U \longrightarrow PG$ is a plot

if and only if $U \times [0, 1] \longrightarrow G: (x, t) \longmapsto c(x)(t)$ is smooth.¹

We remark that Fréchet manifolds embed fully faithfully into diffeological spaces [Los92]. Thus, every Fréchet manifold can be regarded as a diffeological space (the plots are just all smooth maps), and a map between two Fréchet manifolds is smooth in the Fréchet sense if and only if it is smooth in the diffeological sense.

We denote by $PG^{[k]}$ the k -fold fibre product of PG over the evaluation map

$$\text{ev}: PG \longrightarrow G \times G: \gamma \longmapsto (\gamma(0), \gamma(1)),$$

i.e. $PG^{[k]}$ consists of k -tuples of paths all sharing a common initial point and a common end point. Due to the sitting instants, we have a well-defined smooth map

$$\cup: PG^{[2]} \longrightarrow LG: (\gamma_1, \gamma_2) \longmapsto \overline{\gamma_2} \star \gamma_1,$$

where \star denotes the path concatenation, and $\overline{\gamma}$ denotes the reversed path. The set $PG^{[3]}$ is the modelling space for loop fusion: if $(\gamma_1, \gamma_2, \gamma_3) \in PG^{[3]}$, then we have the two loops $\tau_{12} := \gamma_1 \cup \gamma_2$ and $\tau_{23} := \gamma_2 \cup \gamma_3$ which have the common segment γ_2 . Its deletion gives the new loop $\tau_{13} := \gamma_1 \cup \gamma_3$. Loop fusion is multiplicative and strictly associative.

Definition 2.1.

(a) A fusion product on \mathcal{L} is a smooth bundle morphism

$$\lambda: \text{pr}_{12}^* \cup^* \mathcal{L} \otimes \text{pr}_{23}^* \cup^* \mathcal{L} \longrightarrow \text{pr}_{13}^* \cup^* \mathcal{L}$$

over $PG^{[3]}$ that is associative in the sense that

$$\lambda(\lambda(p_{12} \otimes p_{23}) \otimes p_{34}) = \lambda(p_{12} \otimes \lambda(p_{23} \otimes p_{34}))$$

for all $p_{ij} \in \mathcal{L}_{\gamma_i \cup \gamma_j}$ and all $(\gamma_1, \gamma_2, \gamma_3, \gamma_4) \in PG^{[4]}$.

(b) A fusion product λ is called multiplicative if

$$\lambda(p_{12} \otimes p_{23}) \cdot \lambda(p'_{12} \otimes p'_{23}) = \lambda(p_{12} p'_{12} \otimes p_{23} p'_{23})$$

for all elements $p_{ij} \in \mathcal{L}_{\gamma_i \cup \gamma_j}$, $p'_{ij} \in \mathcal{L}_{\gamma'_i \cup \gamma'_j}$ and all $(\gamma_1, \gamma_2, \gamma_3), (\gamma'_1, \gamma'_2, \gamma'_3) \in PG^{[3]}$.

Here $\text{pr}_{ij}: PG^{[3]} \longrightarrow PG^{[2]}$ is the projection to the indexed factors, and \mathcal{L}_τ denotes the fibre of \mathcal{L} over a loop $\tau \in LG$. If \mathcal{L}' is another central extension equipped with a fusion product λ' , then an isomorphism $\varphi: \mathcal{L} \longrightarrow \mathcal{L}'$ is called *fusion-preserving*, if $\varphi(\lambda(p_{12} \otimes p_{23})) = \lambda'(\varphi(p_{12}) \otimes \varphi(p_{23}))$ for all elements $p_{ij} \in \mathcal{L}_{\gamma_i \cup \gamma_j}$ and all $(\gamma_1, \gamma_2, \gamma_3) \in PG^{[3]}$.

A homotopy between loops is the same thing as a path in the loop space: if $\gamma: [0, 1] \longrightarrow LG$ is a path, then

$$h_\gamma: [0, 1] \times S^1 \longrightarrow G: (t, z) \longmapsto \gamma(t)(z)$$

is the corresponding homotopy. The rank of h can at most be two. If it is *less* than two we call the path γ and the homotopy h_γ *thin*. A path γ is thin if and only if $h_\gamma^* \omega = 0$ for all 2-forms $\omega \in \Omega^2(G)$, this leads to the saying that thin homotopies “sweep out a surface of zero area”.

¹A referee suggested to use paths all of whose higher derivatives vanish at the end-points, as opposed to sitting instants, as this would simplify some of the arguments in Sections 3.2 and 3.3. In order to stay consistent with my other papers [Wal12b, Walb, Wal12c], on which Sections 4 and 5 rely on, I have decided to stick to sitting instants.

We denote by $LG_{thin}^2 \subseteq LG \times LG$ the diffeological space consisting of pairs (τ_1, τ_2) of thin homotopic loops, i.e. there exists a homotopy $h : [0, 1] \times S^1 \rightarrow G$ of rank one. The plots are smooth maps $c : U \rightarrow LG_{thin}^2$ such that locally the thin homotopies can be chosen in smooth families, see [Wal12c, Section 3.1].

Definition 2.2.

(a) A thin homotopy equivariant structure on \mathcal{L} is a smooth bundle isomorphism

$$d : \text{pr}_1^* \mathcal{L} \rightarrow \text{pr}_2^* \mathcal{L}$$

over LG_{thin}^2 that satisfies the cocycle condition $d_{\tau_2, \tau_3} \circ d_{\tau_1, \tau_2} = d_{\tau_1, \tau_3}$ for any triple (τ_1, τ_2, τ_3) of thin homotopic loops.

(b) A thin homotopy equivariant structure d is called multiplicative if

$$d_{\tau_0 \gamma_0, \tau_1 \gamma_1}(p \cdot q) = d_{\tau_0, \tau_1}(p) \cdot d_{\gamma_0, \gamma_1}(q)$$

for all $((\tau_0, \gamma_0), (\tau_1, \gamma_1)) \in L(G \times G)_{thin}^2$ and all $p \in \mathcal{L}_{\tau_0}$, $q \in \mathcal{L}_{\gamma_0}$.

Note that $((\tau_0, \gamma_0), (\tau_1, \gamma_1)) \in L(G \times G)_{thin}^2$ means that there exists a thin path (τ, γ) in $L(G \times G)$ connecting (τ_0, γ_0) with (τ_1, γ_1) . It is necessary, but not sufficient, that the paths τ , γ , and $\tau\gamma$ in LG are separately thin.

If \mathcal{L}' is another central extension equipped with a thin homotopy equivariant structure d' , then an isomorphism $\varphi : \mathcal{L} \rightarrow \mathcal{L}'$ is called *thin* if $\varphi(d_{\tau_0, \tau_1}(p)) = d'_{\tau_0, \tau_1}(\varphi(p))$.

A thin homotopy equivariant structure d induces an equivariant structure on \mathcal{L} for the action of the group $\mathcal{D}\text{iff}^+(S^1)$ of orientation-preserving diffeomorphisms of the circle on LG by precomposition. Indeed, suppose φ is an orientation-preserving diffeomorphism, $\tau \in LG$ and $p \in \mathcal{L}_\tau$. Since $\mathcal{D}\text{iff}^+(S^1)$ is connected, there exists a path $\varphi_t \in \mathcal{D}\text{iff}^+(S^1)$ with $\varphi_0 = \text{id}_{S^1}$ and $\varphi_1 = \varphi$. The map $\gamma : [0, 1] \rightarrow LG : t \mapsto \tau \circ \varphi_t$ is a thin path; in particular, τ and $\tau \circ \varphi$ are thin homotopic. We define

$$p \cdot \varphi := d_{\tau, \tau \circ \varphi}(p) \in \mathcal{L}_{\tau \circ \varphi}. \quad (2.1)$$

Lemma 2.3. Formula (2.1) defines a smooth action of $\mathcal{D}\text{iff}^+(S^1)$ on \mathcal{L} . If d is multiplicative, it is an action by group homomorphisms and acts trivially on the central $U(1)$ -subgroup of \mathcal{L} .

Proof. That it is an action follows from the cocycle condition for d . It is smooth because d is a smooth bundle isomorphism over LG_{thin}^2 . If d is multiplicative, we compute for $p_1 \in \mathcal{L}_{\tau_1}$ and $p_2 \in \mathcal{L}_{\tau_2}$

$$(p_1 p_2) \cdot \varphi = d_{\tau_1 \tau_2, (\tau_1 \tau_2) \circ \varphi}(p_1 p_2) = d_{\tau_1 \tau_2, (\tau_1 \circ \varphi)(\tau_2 \circ \varphi)}(p_1 p_2) = d_{\tau_1, \tau_1 \circ \varphi}(p_1) d_{\tau_2, \tau_2 \circ \varphi}(p_2).$$

The restriction of the $\mathcal{D}\text{iff}^+(S^1)$ -action on LG to constant loops is trivial. So if $p \in \mathcal{L}$ projects to a constant loop, we have $p \cdot \varphi = p$. In particular, $\mathcal{D}\text{iff}^+(S^1)$ acts trivially on $U(1)$. \square

Definition 2.4. Suppose \mathcal{L} is equipped with a fusion product λ and a thin homotopy equivariant structure d .

- (a) We say that d is compatible with λ , if for all paths $(\gamma_1, \gamma_2, \gamma_3) \in P(PG^{[3]})$ such that the three paths $t \mapsto \gamma_i(t) \cup \gamma_j(t) \in LG$ are thin, the diagram

$$\begin{array}{ccc} \mathcal{L}_{\gamma_1(0) \cup \gamma_2(0)} \otimes \mathcal{L}_{\gamma_2(0) \cup \gamma_3(0)} & \xrightarrow{\lambda} & \mathcal{L}_{\gamma_1(0) \cup \gamma_3(0)} \\ \downarrow d \otimes d & & \downarrow d \\ \mathcal{L}_{\gamma_1(1) \cup \gamma_2(1)} \otimes \mathcal{L}_{\gamma_2(1) \cup \gamma_3(1)} & \xrightarrow{\lambda} & \mathcal{L}_{\gamma_1(1) \cup \gamma_3(1)} \end{array}$$

is commutative.

- (b) We say that d symmetrizes λ if for all $(\gamma_1, \gamma_2, \gamma_3) \in PG^{[3]}$ and all $p \in \mathcal{L}_{\gamma_1 \cup \gamma_2}$ and $p' \in \mathcal{L}_{\gamma_2 \cup \gamma_3}$

$$d_{\gamma_1 \cup \gamma_3, \overline{\gamma_3} \cup \overline{\gamma_1}}(\lambda(p \otimes p')) = \lambda(d_{\gamma_2 \cup \gamma_3, \overline{\gamma_3} \cup \overline{\gamma_2}}(p') \otimes d_{\gamma_1 \cup \gamma_2, \overline{\gamma_2} \cup \overline{\gamma_1}}(p)).$$

- (c) We say that d is fusive with respect to λ , if it is compatible and symmetrizing,

For (b) we remark that if $r_\pi \in \text{Diff}^+(S^1)$ denotes the rotation by an angle of π , then $(\gamma_i \cup \gamma_j) \circ r_\pi = (\overline{\gamma_j} \cup \overline{\gamma_i})$; in particular, $\gamma_i \cup \gamma_j$ and $\overline{\gamma_j} \cup \overline{\gamma_i}$ are thin homotopic.

Example 2.5. Suppose P is a principal $U(1)$ -bundle over $G \times G$ with connection, such that there exists a connection-preserving isomorphism

$$P_{g_1, g_2} \otimes P_{g_1 g_2, g_3} \cong P_{g_2, g_3} \otimes P_{g_1, g_2 g_3} \quad (2.2)$$

over $G \times G \times G$. The subscript notation is that, for instance, $P_{g_1 g_2, g_3}$ is the pullback of P along the map $(g_1, g_2, g_3) \mapsto (g_1 g_2, g_3)$. The holonomy of P is a smooth map $\eta : LG \times LG \rightarrow U(1)$ such that

$$\eta(\tau_1 \tau_2, \tau_3) \cdot \eta(\tau_1, \tau_2) = \eta(\tau_1, \tau_2 \tau_3) \cdot \eta(\tau_2, \tau_3)$$

for all $\tau_1, \tau_2, \tau_3 \in LG$. Thus, η is a 2-cocycle in the smooth group cohomology of LG . It defines a group structure on $\mathcal{L}_P := U(1) \times LG$ via $(z_1, \tau_1) \cdot (z_2, \tau_2) := (z_1 z_2 \eta(\tau_1, \tau_2), \tau_1 \tau_2)$, making \mathcal{L}_P a central extension of LG . As the holonomy of a bundle, η is a fusion map in the sense of [Wal12b], i.e. it satisfies

$$\eta(\gamma_1 \cup \gamma_3, \gamma'_1 \cup \gamma'_3) = \eta(\gamma_1 \cup \gamma_2, \gamma'_1 \cup \gamma'_2) \cdot \eta(\gamma_2 \cup \gamma_3, \gamma'_2 \cup \gamma'_3).$$

for all $(\gamma_1, \gamma_2, \gamma_3), (\gamma'_1, \gamma'_2, \gamma'_3) \in PG^{[3]}$. This is equivalent to the statement that the trivial fusion product $\lambda((z_{12}, \gamma_1 \cup \gamma_2) \otimes (z_{23}, \gamma_2 \cup \gamma_3)) := (z_{12} z_{23}, \gamma_1 \cup \gamma_3)$ is multiplicative. Likewise, we have the trivial thin homotopy equivariant structure $d(z, \tau_0) := (z, \tau_1)$ for each $(\tau_0, \tau_1) \in LG^2_{thin}$. It is fusive with respect to the trivial fusion product, and it is multiplicative with respect to the group structure defined by η because the holonomy of a connection only depends on the thin homotopy class of a loop.

As a concrete example of this construction, one can take $G = U(1)$ and P the Poincaré bundle over $T := U(1) \times U(1)$, equipped with its canonical connection. In differential cohomology, $P \in \hat{H}^2(T)$ is the cup product of the two projections $\text{pr}_1, \text{pr}_2 : T \rightarrow U(1)$ regarded as elements in $\hat{H}^1(T)$. This implies that the Poincaré bundle has an isomorphism (2.2). Its holonomy can be described in the following way. If $\tau \in LU(1)$, let $n \in \mathbb{Z}$ be the winding number of τ . One can find a smooth map $f : \mathbb{R} \rightarrow \mathbb{R}$ such that $f(t+1) = f(t) + n$ and $\tau = e^{2\pi i f}$. For $\tau = (\tau_1, \tau_2) \in LT$, we get

$$\eta(\tau_1, \tau_2) = \text{Hol}_P(\tau) = \exp 2\pi i \left(n_1 f_2(0) - \int_0^1 f_1(s) f'_2(s) ds \right).$$

Using this formula, we obtain a central extension of $LU(1)$, equipped with a multiplicative fusion product and a multiplicative and fusive thin homotopy equivariant structure.

Example 2.6. Let G be a compact, simple, connected, simply-connected Lie group, so that LG has a universal central extension [PS86]. It can be realized by the following model of Mickelsson [Mic87]. We consider pairs (ϕ, z) where $\phi: D^2 \rightarrow G$ is a smooth map that is radially constant near the boundary, and $z \in U(1)$. We impose the following equivalence relation:

$$(\phi, z) \sim (\phi', z') \iff \partial\phi = \partial\phi' \quad \text{and} \quad z = z' \cdot e^{2\pi i S_{\text{WZ}}(\Phi)}.$$

Here, $\partial\phi \in LG$ denotes the restriction of ϕ to the boundary, and $\Phi: S^2 \rightarrow G$ is the map defined on the northern hemisphere by ϕ (with the orientation-preserving identification) and on the southern hemisphere by ϕ' (with the orientation-reversing identification). The symbol S_{WZ} stands for the *Wess-Zumino term* defined as follows. Because G is 2-connected, the map Φ can be extended to a smooth map $\tilde{\Phi}: D^3 \rightarrow G$ defined on the solid ball. Then,

$$S_{\text{WZ}}(\Phi) := \int_{D^3} \tilde{\Phi}^* H \quad \text{with} \quad H := \frac{1}{6} \langle \theta \wedge [\theta \wedge \theta] \rangle \in \Omega^3(G). \quad (2.3)$$

Here, $\theta \in \Omega^1(G, \mathfrak{g})$ is the left-invariant Maurer-Cartan form on G . The bilinear form $\langle -, - \rangle$ is normalized such that the closed 3-form H represents a generator $H^3(G, \mathbb{Z}) \cong \mathbb{Z}$. Now, the total space of the principal $U(1)$ -bundle \mathcal{L}_G is the set of equivalence classes of pairs (ϕ, z) . The bundle projection sends (ϕ, z) to $\partial\phi \in LG$, and the $U(1)$ -action is given by multiplication in the $U(1)$ -component. The group structure on \mathcal{L}_G turning it into a central extension is given by the *Mickelsson product* [Mic87]:

$$\mathcal{L}_G \times \mathcal{L}_G \rightarrow \mathcal{L}_G: ((\phi_1, z_1), (\phi_2, z_2)) \mapsto (\phi_1 \phi_2, z_1 z_2 \cdot \exp 2\pi i \left(- \int_{D^2} (\phi_1, \phi_2)^* \rho \right)),$$

where ρ is defined by

$$\rho := \frac{1}{2} \langle \text{pr}_1^* \theta \wedge \text{pr}_2^* \bar{\theta} \rangle \in \Omega^2(G \times G). \quad (2.4)$$

The two differential forms H and ρ satisfy the identities

$$\Delta H := H_{g_1} - H_{g_1 g_2} + H_{g_2} = d\rho \quad (2.5)$$

$$\Delta \rho := \rho_{g_1, g_2} + \rho_{g_1 g_2, g_3} - \rho_{g_2, g_3} - \rho_{g_1, g_2 g_3} = 0. \quad (2.6)$$

for all $g_1, g_2, g_3 \in G$, where the subscripts are meant so that $\rho_{g_1 g_2, g_3}$ is the pullback of ρ along the map $(g_1, g_2, g_3) \mapsto (g_1 g_2, g_3)$. Eq. (2.5) assures that the Mickelsson product is well-defined on equivalence classes, and (2.6) implies its associativity.

A fusion product on \mathcal{L}_G is defined as follows. For $(\gamma_1, \gamma_2, \gamma_3) \in PG^{[3]}$, we define

$$\begin{aligned} \lambda_{\gamma_1, \gamma_2, \gamma_3} : \mathcal{L}_G|_{\gamma_1 \cup \gamma_2} \otimes \mathcal{L}_G|_{\gamma_2 \cup \gamma_3} &\rightarrow \mathcal{L}_G|_{\gamma_1 \cup \gamma_3} \\ (\phi_{12}, z_{12}) \otimes (\phi_{23}, z_{23}) &\mapsto (\phi_{13}, z_{12} z_{23} \cdot e^{-2\pi i S_{\text{WZ}}(\Psi)}), \end{aligned} \quad (2.7)$$

where $\phi_{13}: D^2 \rightarrow G$ is an arbitrarily chosen smooth map with $\partial\phi_{13} = \gamma_1 \cup \gamma_3$, and $\Psi: S^2 \rightarrow G$ is obtained by trisecting S^2 along the longitudes $0, \frac{2\pi}{3}$ and $\frac{4\pi}{3}$, and prescribing Ψ on each sector with the maps ϕ_{12}, ϕ_{23} (with orientation-preserving identification) and ϕ_{13} (with orientation-reversing identification), respectively, see Figure 1. That (2.7) is independent of the choice of ϕ_{13} follows from the identity $S_{\text{WZ}}(\Psi) = S_{\text{WZ}}(\Psi') S_{\text{WZ}}(\Phi_{13})$ for Wess-Zumino terms, where Ψ' is obtained as described above but using a different map ϕ'_{13} instead of ϕ_{13} , and Φ_{13} is obtained in the way described earlier from ϕ_{13} and ϕ'_{13} . Definition (2.7) is also well-defined under the equivalence relation \sim due to a similar identity for Wess-Zumino terms. Associativity follows from reparameterization invariance of the integral, and multiplicativity follows from the Polyakov-Wiegmann formula

$$e^{2\pi i S_{\text{WZ}}(\Phi_1)} \cdot e^{2\pi i S_{\text{WZ}}(\Phi_2)} = e^{2\pi i S_{\text{WZ}}(\Phi_1 \Phi_2)} \cdot \exp 2\pi i \left(\int_{(\Phi_1, \Phi_2)} \rho \right), \quad (2.8)$$

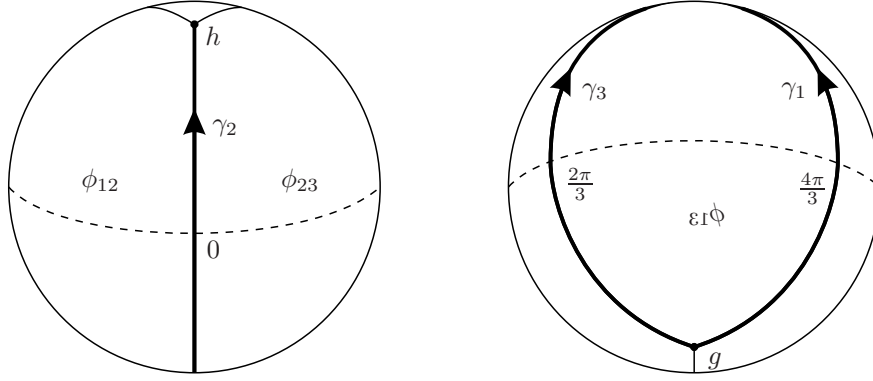


Figure 1: Front and back view to the map Ψ in the definition of the fusion product over a triple $(\gamma_1, \gamma_2, \gamma_3) \in PG^{[3]}$ with common initial point g and common end point h .

which in turn follows from (2.6).

A thin homotopy equivariant structure d is defined as follows. Suppose $(\gamma_0, \gamma_1) \in LG_{thin}^2$. Then,

$$d_{\gamma_0, \gamma_1} : \mathcal{L}_G|_{\gamma_0} \longrightarrow \mathcal{L}_G|_{\gamma_1} : (\phi_0, z_0) \longmapsto (\phi_1, z_0 \cdot e^{2\pi i S_{WZ}(\Phi_\gamma)}),$$

where $\phi_1 : D^2 \longrightarrow G$ is an arbitrarily chosen smooth map with $\partial\phi_1 = \gamma_1$, and $\Phi_\gamma : S^2 \longrightarrow G$ is the following map. On the polar caps $D^2 \subseteq S^2$ we prescribe Φ with ϕ_0 (around the north pole with orientation-reversing identification) and ϕ_1 (around the south pole with orientation-preserving identification), and on the remaining cylinder $Z \cong [0, 1] \times S^1$ by the homotopy h_γ of a thin path $\gamma : [0, 1] \longrightarrow LG$ with $\gamma_0 = \gamma(0)$ and $\gamma_1 = \gamma(1)$. A different choice of ϕ_1 gives an equivalent result. If another thin path γ' is chosen, then the two paths constitute a rank one loop in LG , i.e. a map $\Phi : S^1 \times S^1 \longrightarrow G$ of rank one, and we have to prove that $S_{WZ}(\Phi) = 0$. This follows as a special case of [Walb, Prop. 3.3.1]. In the next paragraphs we give an independent argument for the vanishing of Wess-Zumino terms for maps $\Phi : \Sigma \longrightarrow G$ of rank one, mapping a compact oriented surface Σ to a connected, simple, simply-connected compact Lie group G .

Let T be a maximal torus of G , \mathfrak{t} its Lie algebra, and $\mathfrak{A} \subseteq \mathfrak{t}$ a closed Weyl alcove, a simplex with vertices $0 = \mu_0, \dots, \mu_r$, where r is the rank of G . We let $F_0 \subseteq \mathfrak{A}$ denote the closed face spanned by $\{\mu_1, \dots, \mu_r\}$, $\mathfrak{A}_0 := \mathfrak{A} \setminus F_0$ its complement, and

$$U := \{h \exp \xi h^{-1} \mid h \in G \text{ and } \xi \in \mathfrak{A}_0\}$$

the corresponding open subset of G , which deformation retracts to $1 \in G$, see [Mei02, Lemma 5.1]. We show below that there exists $g \in G$ such that the left-translated map $\Phi_g : \Sigma \longrightarrow G$, $\Phi_g(x) := g\Phi(x)$, has its image contained in U . Due to the left-invariance of H , we have $S_{WZ}(\Phi) = S_{WZ}(\Phi_g)$. By composition with the deformation retract one obtains a rank-two map $\tilde{\Phi}_g : D \longrightarrow U$ defined on a 3-dimensional manifold D with $\partial D = \Sigma$, such that $\tilde{\Phi}_g|_\Sigma = \Phi_g$. Thus,

$$S_{WZ}(\Phi) = S_{WZ}(\Phi_g) = \int_D \tilde{\Phi}_g^* H = 0.$$

Next we prove the existence of an appropriate left translating element g .

Recall that $g \in G$ is called *regular* if the conjugacy class of g has maximal dimension. For the class of Lie groups under consideration, an element is regular if and only if it is conjugate to $\exp \xi$

for ξ in the interior of \mathfrak{A} . In particular, the open subset G^{reg} of regular elements is contained in U . We define $Q := G \setminus U$, and obtain $Q \subseteq G \setminus G^{\text{reg}}$. The set $G \setminus G^{\text{reg}}$ is the disjoint union of finitely many submanifolds of G of codimension ≥ 2 [DK00, Page 137 Item (g)]. Let Q_α be one of these submanifolds. We consider the family $g \mapsto \Phi_g$ of left translates, which, as a map $G \times \Sigma \rightarrow G$, is a submersion and hence transverse to Q_α . By the parametric transversality theorem [Hir76, Chp. 3, Thm. 2.7] the set $G_\alpha := \{g \in G \mid \Phi_g \text{ is transverse to } Q_\alpha\}$ is residual. But for dimensional reasons, a rank one map can only be transverse to a submanifold of codimension ≥ 2 if its image does not intersect Q_α , i.e. $G_\alpha = \{g \in G \mid \Phi_g(\Sigma) \cap Q_\alpha = \emptyset\}$. The intersection of finitely many residual sets is residual, in particular non-empty. Thus, there exists $g \in G$ such that $\Phi_g(\Sigma)$ does not intersect any of the submanifolds Q_α , i.e. $\Phi_g(\Sigma) \subseteq U$. This finishes the proof that $S_{\text{WZ}}(\Phi) = 0$.

By now we have shown that the thin homotopy equivariant structure d on \mathcal{L}_G is well-defined as a bundle isomorphism over LG_{thin}^2 . The cocycle condition follows from the identity $S_{\text{WZ}}(\Phi_{\gamma_2}) + S_{\text{WZ}}(\Phi_{\gamma_1}) = S_{\text{WZ}}(\Phi_{\gamma_2 \star \gamma_1})$, if at $\gamma_1(1) = \gamma_2(0)$ the same extension $\phi : D^2 \rightarrow G$ is chosen. Multiplicativity follows because in the Polyakov-Wiegmann formula (2.8) the error term vanishes, as the integral of the 2-form ρ along a thin path (γ, τ) through $L(G \times G)$ gives zero. Compatibility with the fusion product λ can be seen by inspection of the occurring integrals. Finally, let us check in some more detail that d symmetrizes λ . For $(\gamma_1, \gamma_2, \gamma_3) \in PG^{[3]}$, let $\phi_{12}, \phi_{23}, \phi_{13} : D^2 \rightarrow G$ such that $\lambda((\phi_{12}, 1) \otimes (\phi_{23}, 1)) = (\phi_{13}, 1)$ holds. This means that $e^{S_{\text{WZ}}(\Psi)} = 1$, with $\Psi : S^2 \rightarrow G$ defined as shown in Figure 1. We have to check that

$$\lambda((\phi_{23}, 1) \cdot r_\pi \otimes (\phi_{12}, 1) \cdot r_\pi) = (\phi_{13}, 1) \cdot r_\pi, \quad (2.9)$$

where $r_\pi \in \mathcal{D}\text{iff}^+(S^1)$ is the rotation by an angle of π . The definition of the thin homotopy equivariant structure implies $(\phi, z) \cdot r_\pi = (\phi \circ r_\pi, z)$, where on the right hand side r_π is extended to a rotation of D^2 . In order to check (2.9) we have to form the map $\Psi' : S^2 \rightarrow G$ using $\phi_{23} \circ r_\pi$, $\phi_{12} \circ r_\pi$ and $\phi_{13} \circ r_\pi$. By inspection, $\Psi' = \Psi \circ r_\pi$, where now r_π is extended to a rotation of S^2 about the front-back axis, compare Figures 1 and 2. Thus, $e^{S_{\text{WZ}}(\Psi)} = 1$ and (2.9) holds.

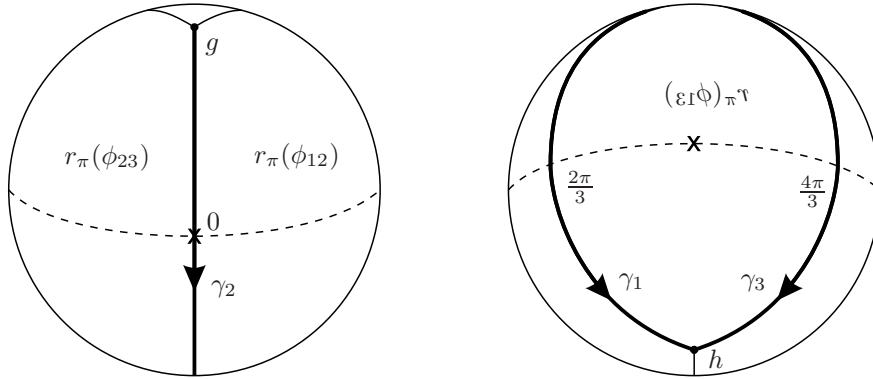


Figure 2: Front and back view to the map Ψ' defined as in Figure 1 but using the maps $\phi_{23} \circ r_\pi$, $\phi_{12} \circ r_\pi$ and $\phi_{13} \circ r_\pi$ instead of ϕ_{12} , ϕ_{23} and ϕ_{13} . The pictures are precisely the ones of Figure 1 rotated by an angle of π around the axis that intersects S^2 at the two marked points.

3 Features of fusion and thin homotopy equivariance

In this section we derive some consequences of the presence of a fusion product and a thin homotopy equivariant structure on a central extension. In particular, all results of this section hold for thin fusion extensions, and thus, by our main theorem, for transgressive central extensions.

3.1 Flat loops and retraction

Loops in the image of the map $b : PG \longrightarrow LG : \gamma \longmapsto \gamma \cup \gamma$ are called *flat loops*. Note that b is a group homomorphism and that every constant loop is flat. Suppose ϕ is a smoothing map, i.e. it is a smooth map $\phi : [0, 1] \longrightarrow [0, 1]$ with $\phi(0) = 0$ and $\phi(1) = 1$, locally constant in a neighborhood of $\{0, 1\}$, and smoothly homotopic to $\text{id}_{[0, 1]}$. *Path retraction* is the map

$$[0, 1] \times PG \longrightarrow PG : (t, \gamma) \longmapsto \phi_\gamma(t)$$

defined by $\phi_\gamma(t)(s) := \gamma(t\phi(s))$; ϕ is necessary to guarantee that $\phi_\gamma(t)$ has sitting instants. Clearly, $t \longmapsto b(\phi_\gamma(t))$ is a thin path in LG ; in particular, for every $t \in [0, 1]$, $b(\phi_\gamma(t))$ is thin homotopic to $b(\gamma \circ \phi)$, which is in turn thin homotopic to $b(\gamma)$.

Proposition 3.1.1. *Suppose \mathcal{L} is a central extension of LG equipped with a multiplicative fusion product λ and a fusive thin homotopy equivariant structure d . Then, there exists a unique section $PG \longrightarrow \mathcal{L} : \gamma \longmapsto 1_\gamma$ along b such that $\lambda(1_\gamma, 1_\gamma) = 1_\gamma$. It has the following properties:*

- (i) *It is neutral with respect to fusion, i.e. $\lambda(p \otimes 1_{\gamma_2}) = p = \lambda(1_{\gamma_1} \otimes p)$ for all $p \in \mathcal{L}_{\gamma_1 \cup \gamma_2}$.*
- (ii) *It is a group homomorphism, i.e. $1_{\gamma_1} \cdot 1_{\gamma_2} = 1_{\gamma_1 \gamma_2}$.*
- (iii) *It is retraction-invariant, i.e. $1_{\phi_\gamma(t)} = d_{b(\gamma), b(\phi_\gamma(t))}(1_\gamma)$ for all $t \in [0, 1]$.*

Proof. Two sections $s, s' : PG \longrightarrow \mathcal{L}$ differ by a smooth map $\alpha : PG \longrightarrow U(1)$, i.e., $s' = \alpha s$. If the sections satisfy $\lambda(s, s) = s$ and $\lambda(s', s') = s'$, we get $\alpha^2 = \alpha$ and so $s = s'$. For the existence, we notice that pulling back λ along the diagonal map $PG \longrightarrow PG^{[3]}$ shows that $b^*\mathcal{L}$ is trivializable. Let $s : PG \longrightarrow \mathcal{L}$ be any section. Then, there exists a unique smooth map $\alpha : PG \longrightarrow U(1)$ such that $\lambda(s \otimes s) = s \cdot \alpha$. Hence, $1_\gamma := s(\gamma) \cdot \alpha(\gamma)$ has the desired property.

For (i) we have $\lambda(1_{\gamma_1} \otimes p) = \lambda(\lambda(1_{\gamma_1} \otimes 1_{\gamma_1}) \otimes p) = \lambda(1_{\gamma_1} \otimes \lambda(1_{\gamma_1} \otimes p))$ using the associativity of the fusion product, and since $\lambda(1_{\gamma_1} \otimes -)$ is an isomorphism, we get $p = \lambda(1_{\gamma_1} \otimes p)$. Analogously we show neutrality from the right. For (ii) we have

$$\lambda((1_{\gamma_1} \cdot 1_{\gamma_2}) \otimes (1_{\gamma_1} \cdot 1_{\gamma_2})) = \lambda(1_{\gamma_1} \otimes 1_{\gamma_1}) \cdot \lambda(1_{\gamma_2} \otimes 1_{\gamma_2}) = 1_{\gamma_1} \cdot 1_{\gamma_2}$$

using the multiplicativity of the fusion product; the uniqueness of the section then shows that $1_{\gamma_1} \cdot 1_{\gamma_2} = 1_{\gamma_1 \gamma_2}$. For (iii) we compute with the compatibility of Definition 2.4 (a)

$$\lambda(d_{b(\gamma), b(\phi_\gamma(t))}(1_\gamma) \otimes d_{b(\gamma), b(\phi_\gamma(t))}(1_\gamma)) = d_{b(\gamma), b(\phi_\gamma(t))}(\lambda(1_\gamma \otimes 1_\gamma)) = d_{b(\gamma), b(\phi_\gamma(t))}(1_\gamma),$$

from which the claim follows again from the uniqueness of the section. \square

In particular, the restriction of \mathcal{L} to flat loops is canonically trivializable as a central extension of PG by $U(1)$.

3.2 Loop concatenation

Let \mathcal{L} be a central extension of LG equipped with a fusion product. We start with the prototypical situation for loop concatenation: let $\gamma_1, \gamma_2 \in PG$ be closed, i.e. loops with sitting instants. We have $(\gamma_1, \text{id}, \gamma_2) \in PG^{[3]}$. The concatenated loop is $\gamma_1 \cup \gamma_2$. If $p_1 \in \mathcal{L}_{\gamma_1 \cup \text{id}}$ and $p_2 \in \mathcal{L}_{\text{id} \cup \gamma_2}$, then

$$\lambda(p_1 \otimes p_2) \in \mathcal{L}_{\gamma_1 \cup \gamma_2};$$

this lifts loop concatenation from LG to \mathcal{L} . In the following we use a fusive thin homotopy equivariant structure in order to generalize to arbitrary loops that admit concatenation (not necessarily with sitting instants).

We denote by $LG \times_G^\infty LG$ the subset of $LG \times LG$ consisting of pairs (τ_1, τ_2) such that $\tau_1(1) = \tau_2(1)$ and the concatenation $\overline{\tau_2} \star \tau_1$ is again a smooth loop. Thus, we have a well-defined map

$$\text{con} : LG \times_G^\infty LG \longrightarrow LG : (\tau_1, \tau_2) \longmapsto \overline{\tau_2} \star \tau_1.$$

If we equip $LG \times_G^\infty LG \subseteq LG \times LG$ with the subspace diffeology (i.e. a map $c : U \longrightarrow LG \times_G^\infty LG$ is a plot if and only if its extension to $LG \times LG$ is smooth), then con is smooth. Further, con is a group homomorphism.

We fix a smoothing map ϕ and construct new smooth maps $\phi_1, \phi_2 : [0, 1] \longrightarrow [0, 1]$ by setting:

$$\phi_1(t) := \begin{cases} 0 & 0 \leq t \leq \frac{1}{2} \\ \phi(2t-1) & \frac{1}{2} \leq t \leq 1 \end{cases} \quad \text{and} \quad \phi_2(t) := \begin{cases} \phi(2t) & 0 \leq t \leq \frac{1}{2} \\ 1 & \frac{1}{2} \leq t \leq 1. \end{cases}$$

These cover well-defined smooth maps $\phi_1, \phi_2 : S^1 \longrightarrow S^1$ that are smoothly homotopic to id_{S^1} . Thus, if $(\tau_1, \tau_2) \in LG \times_G^\infty LG$ with $g := \tau_1(1) = \tau_2(1)$, then $\tilde{\tau}_k := \tau_k \circ \phi_k$ is thin homotopic to τ_k for $k = 1, 2$; furthermore $\tilde{\tau}_1 = (\tau_1 \circ \phi) \cup \text{id}_g$ and $\tilde{\tau}_2 = \text{id}_g \cup (\tau_2 \circ \phi)$.

Suppose we have elements $p_k \in \mathcal{L}_{\tau_k}$ for $k = 1, 2$. Using the thin homotopy equivariant structure, we define $\tilde{p}_k := d_{\tau_k, \tilde{\tau}_k}(p_k)$, and form the fusion product

$$\tilde{p} := \lambda(\tilde{p}_1 \otimes \tilde{p}_2) \in \mathcal{L}_{(\tau_1 \circ \phi) \cup (\tau_2 \circ \phi)}.$$

The loop $(\tau_1 \circ \phi) \cup (\tau_2 \circ \phi)$ is thin homotopic to $\text{con}(\tau_1, \tau_2)$. We obtain an element

$$p := d_{(\tau_1 \circ \phi) \cup (\tau_2 \circ \phi), \text{con}(\tau_1, \tau_2)}(\tilde{p}) \in \mathcal{L}_{\text{con}(\tau_1, \tau_2)}.$$

Proposition 3.2.1. *Suppose \mathcal{L} is a central extension of LG equipped with a fusion product λ and a fusive thin homotopy equivariant structure d . Then, the assignment $(p_1, p_2) \longmapsto p$ defined above is a smooth map*

$$\widetilde{\text{con}} : \text{pr}_1^* \mathcal{L} \otimes \text{pr}_2^* \mathcal{L} \longrightarrow \text{con}^* \mathcal{L}$$

over $LG \times_G^\infty LG$ that is independent of the choice of the smoothing function. If λ and d are multiplicative, then

$$\widetilde{\text{con}}(p_1 \otimes p_2) \cdot \widetilde{\text{con}}(p'_1 \otimes p'_2) = \widetilde{\text{con}}(p_1 p'_1 \otimes p_2 p'_2)$$

for all $p_i \in \mathcal{L}_{\tau_i}$, $p'_i \in \mathcal{L}_{\tau'_i}$ with $i = 1, 2$ and $(\tau_1, \tau_2), (\tau'_1, \tau'_2) \in LG \times_G^\infty LG$.

Proof. We show the independence of the smoothing function ϕ . If ϕ' is another smoothing function, then ϕ and ϕ' are smoothly homotopic. We obtain loops $\tilde{\tau}'_1$ and $\tilde{\tau}'_2$ that are thin homotopic to $\tilde{\tau}_1$ and

$\tilde{\tau}_2$, respectively. The diagram

$$\begin{array}{ccccc}
& & \mathcal{L}_{\tilde{\tau}_1} \otimes \mathcal{L}_{\tilde{\tau}_2} & \xrightarrow{\lambda} & \mathcal{L}_{(\tau_1 \circ \phi) \cup (\tau_2 \circ \phi)} \\
& \nearrow d & \downarrow d & & \downarrow d \\
\mathcal{L}_{\tau_1} \otimes \mathcal{L}_{\tau_2} & & & & \mathcal{L}_{con(\tau_1, \tau_2)} \\
& \searrow d & \downarrow d & & \downarrow d \\
& & \mathcal{L}_{\tilde{\tau}'_1} \otimes \mathcal{L}_{\tilde{\tau}'_2} & \xrightarrow{\lambda} & \mathcal{L}_{(\tau_1 \circ \phi') \cup (\tau_2 \circ \phi')}
\end{array}$$

is commutative: the triangular diagrams commute due to the cocycle condition for d (Definition 2.2 (a)), and the rectangular diagram commutes due to the compatibility condition for λ and d (Definition 2.4 (a)). This shows the independence of the choice of ϕ .

Smoothness can be checked with fixed smoothing function and then follows from the smoothness of the thin structure and the fusion product. In order to see the multiplicativity, we notice that the thin paths from τ_i to $\tilde{\tau}_i$ give a thin path in $L(G \times G)$. Thus, the multiplicativity of d and the one of the fusion product imply the claimed property. \square

3.3 Disjoint-commutativity

A loop $\tau \in LG$ is *supported on an interval* $I \subseteq S^1$ if $\tau(z) = 1$ for all $z \in S^1 \setminus I$.

Theorem 3.3.1. *Suppose \mathcal{L} is a central extension of LG equipped with a multiplicative fusion product and a multiplicative, fusive thin homotopy equivariant structure. Then, \mathcal{L} is disjoint commutative, i.e. if $\tau_1, \tau_2 \in LG$ are supported on disjoint intervals, and $p_1 \in \mathcal{L}_{\tau_1}$ and $p_2 \in \mathcal{L}_{\tau_2}$, then $p_1 \cdot p_2 = p_2 \cdot p_1$.*

Theorem 3.3.1 applies to the universal central extension \mathcal{L}_G of a compact, simple, connected, simply-connected Lie group G , as seen in Example 2.6. In that case, disjoint commutativity was known to hold, verified by a direct calculation using the Mickelsson model [GF93, Lemma 3.1].

For the proof Theorem 3.3.1, we start with the following prototypical situation. Suppose $\tau_1, \tau_2 \in PG$ are loops based at $1 \in G$. Then, $\tau_1 \cup \text{id}_1$ and $\text{id}_1 \cup \tau_2$ are loops supported on disjoint intervals. In particular,

$$(\tau_1 \cup \text{id}_1)(\text{id}_1 \cup \tau_2) = (\text{id}_1 \cup \tau_2)(\tau_1 \cup \text{id}_1).$$

Proposition 3.3.2. *We have $p_1 \cdot p_2 = p_2 \cdot p_1$ for all $p_1 \in \mathcal{L}_{\tau_1 \cup \text{id}_1}$ and $p_2 \in \mathcal{L}_{\text{id}_1 \cup \tau_2}$.*

In order to prove this, we note that the multiplication in this particular situation coincides with the fusion product.

Lemma 3.3.3. $p_1 \cdot p_2 = \lambda(p_1 \otimes p_2)$.

Proof. Proposition 3.1.1 (ii) implies $1_{\text{id}_1} = 1$. Thus, over the triple $(\tau_1, \text{id}_1, \text{id}_1) \in PG^{[3]}$ we can write $p_1 = \lambda(p_1 \otimes 1)$. Likewise, over $(\text{id}_1, \text{id}_1, \tau_2)$ we have $p_2 = \lambda(1 \otimes p_2)$. Multiplicativity of the fusion product then shows the claim:

$$p_1 \cdot p_2 = \lambda(p_1 \otimes 1) \cdot \lambda(1 \otimes p_2) = \lambda(p_1 \cdot 1 \otimes 1 \cdot p_2) = \lambda(p_1 \otimes p_2).$$

\square

Proof of Proposition 3.3.2. That the thin homotopy equivariant structure symmetrizes the fusion product (Definition 2.4 (b)) means, over the triple $(\tau_1, \text{id}_1, \tau_3)$,

$$d_{\tau_1 \cup \tau_3, \overline{\tau_3} \cup \overline{\tau_1}}(\lambda(p_1 \otimes p_2)) = \lambda(d_{\text{id}_1 \cup \tau_3, \overline{\tau_3} \cup \text{id}_1}(p_2) \otimes d_{\tau_1 \cup \text{id}_1, \text{id}_1 \cup \overline{\tau_1}}(p_1)).$$

We can apply Lemma 3.3.3 on the left (to the given data) and on the right (to the pair $(\overline{\tau_3}, \overline{\tau_1})$ and the elements $d_{\text{id}_1 \cup \tau_3, \overline{\tau_3} \cup \text{id}_1}(p_2)$ and $d_{\tau_1 \cup \text{id}_1, \text{id}_1 \cup \overline{\tau_1}}(p_1)$) and obtain

$$d_{\tau_1 \cup \tau_3, \overline{\tau_3} \cup \overline{\tau_1}}(p_1 \cdot p_2) = d_{\text{id}_1 \cup \tau_3, \overline{\tau_3} \cup \text{id}_1}(p_2) \cdot d_{\tau_1 \cup \text{id}_1, \text{id}_1 \cup \overline{\tau_1}}(p_1).$$

Now we want to use the multiplicativity of the thin homotopy equivariant structure. We claim that $((\text{id}_1 \cup \tau_3, \tau_1 \cup \text{id}_1), (\overline{\tau_3} \cup \text{id}_1, \text{id}_1 \cup \overline{\tau_1})) \in L(G \times G)_{\text{thin}}^2$. Indeed, let $r_{-\pi} : S^1 \rightarrow S^1$ denote the rotation by an angle of $-\pi$. Then, $(\text{id}_1 \cup \tau_3) \circ r_{-\pi} = \overline{\tau_3} \cup \text{id}_1$ and $(\tau_1 \cup \text{id}_1) \circ r_{-\pi} = \text{id}_1 \cup \overline{\tau_1}$. Fixing a path φ_t from $\varphi_0 = \text{id}_{S^1}$ to $\varphi_1 = r_{-\pi}$ we obtain a path

$$((\text{id}_1 \cup \tau_3) \circ \varphi_t, (\tau_1 \cup \text{id}_1) \circ \varphi_t)$$

in $L(G \times G)$, which is thin as it factors through $\varphi_t(z) \in S^1$. Thus, the multiplicativity of the thin structure now implies $p_1 \cdot p_2 = p_2 \cdot p_1$. \square

Proof of Theorem 3.3.1. We consider two general loops $\tau_1, \tau_2 \in LG$ with disjoint supports $I, J \subseteq S^1$. There is an orientation-preserving diffeomorphism $\varphi : S^1 \rightarrow S^1$ such that $\varphi^{-1}(I) \subseteq (\frac{1}{2}, 1)$ and $\varphi^{-1}(J) \subseteq (0, \frac{1}{2})$. Then, $\tau_1 \circ \varphi = \tau'_1 \cup \text{id}_1$ and $\tau_2 \circ \varphi = \text{id}_1 \cup \tau'_2$, where $\tau'_1(e^{2\pi i t}) = (\tau_1 \circ \varphi)(e^{\pi i(1-t)})$ and $\tau'_2(e^{2\pi i t}) = (\tau_2 \circ \varphi)(e^{\pi i t})$. The map φ is homotopic to id_{S^1} via a family φ_t with $\varphi_0 = \text{id}_{S^1}$ and $\varphi_1 = \varphi$. The path $(\tau_1 \circ \varphi_t, \tau_2 \circ \varphi_t) = (\tau_1, \tau_2) \circ \varphi_t$ in $L(G \times G)$ is thin. If $p_1 \in \mathcal{L}_{\tau_1}$ and $p_2 \in \mathcal{L}_{\tau_2}$, then $d_{\tau_1, \tau'_1 \cup \text{id}_1}(p_1) \in \mathcal{L}_{\tau'_1 \cup \text{id}_1}$ and $d_{\tau_2, \text{id}_1 \cup \tau'_2}(p_2) \in \mathcal{L}_{\text{id}_1 \cup \tau'_2}$. Proposition 3.3.2 applies and

$$d_{\tau_1, \tau'_1 \cup \text{id}_1}(p_1) \cdot d_{\tau_2, \text{id}_1 \cup \tau'_2}(p_2) = d_{\tau_2, \text{id}_1 \cup \tau'_2}(p_2) \cdot d_{\tau_1, \tau'_1 \cup \text{id}_1}(p_1).$$

We use the multiplicativity of the thin structure: on the left for the thin path $(\tau_1, \tau_2) \circ \varphi_t$ and on the right for the thin path $(\tau_2, \tau_1) \circ \varphi_t$, and obtain

$$d_{\tau_1 \tau_2, (\tau'_1 \cup \text{id}_1)(\text{id}_1 \cup \tau'_2)}(p_1 \cdot p_2) = d_{\tau_2 \tau_1, (\text{id}_1 \cup \tau'_2)(\tau'_1 \cup \text{id}_1)}(p_2 \cdot p_1).$$

We have

$$(\tau'_1 \cup \text{id}_1)(\text{id}_1 \cup \tau'_2) = (\text{id}_1 \cup \tau'_2) \cdot (\tau'_1 \cup \text{id}_1) = \tau'_1 \cup \tau'_2 \quad \text{and} \quad \tau_1 \tau_2 = \tau_2 \tau_1,$$

and as $d_{\tau_1 \tau_2, \tau'_1 \cup \tau'_2}$ is an isomorphism, we get $p_1 \cdot p_2 = p_2 \cdot p_1$. \square

4 Integrable thin homotopy equivariant structures

A thin homotopy equivariant structure on a central extension \mathcal{L} can be induced from certain connections on (the underlying principal $U(1)$ -bundle of) \mathcal{L} .

Definition 4.1.

- (a) A connection ν on \mathcal{L} is called thin, if two thin paths γ_1, γ_2 with common initial point $\gamma_1(0) = \gamma_2(0)$ and common end point $\gamma_1(1) = \gamma_2(1)$ induce the same parallel transport, $pt_{\gamma_1}^\nu = pt_{\gamma_2}^\nu$.
- (b) A thin connection ν on \mathcal{L} is called superficial, if two loops $\tau_1, \tau_2 \in LLG$ have the same holonomy whenever they are homotopic via a map $h : [0, 1] \times S^1 \times S^1 \rightarrow G$ of rank at most two.

Every thin connection ν induces a thin homotopy equivariant structure d^ν by $d_{\tau_0, \tau_1}^\nu := pt_\tau^\nu$, where τ is an arbitrary thin path from τ_0 to τ_1 , see [Wal12c, Lemma 3.1.5]. Superficiality is a property of connections in the image of transgression, and will be relevant in Section 5.

Definition 4.2. Suppose \mathcal{L} is equipped with a fusion product λ and a thin connection ν .

(a) ν is compatible with λ , if the bundle morphism λ over $PG^{[3]}$ is connection-preserving.

(b) ν symmetrizes λ , if d^ν is symmetrizing.

(c) ν is fusive if it is compatible and symmetrizing.

If ν is a thin and fusive connection, then d^ν is fusive with respect to λ . The question whether or not a given thin homotopy equivariant structure can be induced from a thin connection gives rise to the following definition.

Definition 4.3. A thin homotopy equivariant structure d is called thin structure, if there exists a superficial connection ν such that $d^\nu = d$. In the presence of a fusion product, it is called fusive thin structure if ν can be chosen fusive.

It remains to discuss multiplicativity in the setting of thin structures. This requires some attention, because it is not clear which multiplicativity condition one should impose on a connection ν . First of all we note that every connection ν on \mathcal{L} determines a 1-form $\epsilon_\nu \in \Omega^1(LG \times LG)$ by

$$\nu_{p_1}(X_1) + \nu_{p_2}(X_2) = \nu_{p_1 p_2}(p_1 X_2 + X_1 p_1) + \epsilon_\nu|_{\tau_1, \tau_2}(\mathcal{X}_1, \mathcal{X}_2) \quad (4.1)$$

for all $\tau_1, \tau_2 \in LG$, $\mathcal{X}_i \in T_{\tau_i} LG$, as well as $p_i \in \mathcal{L}_{\tau_i}$ and $X_i \in T_{p_i} \mathcal{L}$ such that $p_*(X_i) = \mathcal{X}_i$. We call ϵ_ν the *error 1-form* of ν , it can be seen as a measure for the non-multiplicativity of ν . We want to impose a multiplicativity condition for the connection ν by requiring that ϵ_ν admits a path splitting in the following sense.

Definition 4.4. Let X be a smooth manifold, $k \in \mathbb{N}$, and $\epsilon \in \Omega^k(LX)$. A path splitting of ϵ is a k -form $\kappa \in \Omega^k(PX)$ such that $\cup^* \epsilon = \text{pr}_2^* \kappa - \text{pr}_1^* \kappa$ on $PX^{[2]}$.

We formulate two additional conditions for path splittings that will be required, too. We recall that for every (Fréchet or diffeological) Lie group K and every $k \in \mathbb{N}$ we have a complex

$$0 \longrightarrow \Omega^k(K) \longrightarrow \Omega^k(K \times K) \longrightarrow \dots \quad (4.2)$$

whose differential Δ is the alternating sum over the pullbacks along the face maps of the simplicial manifold BK . A form in the kernel of Δ is called *multiplicative*.

For $K = LG$, (4.1) implies that ϵ_ν is multiplicative: $\Delta \epsilon_\nu = 0$.

For $K = PG$, it makes sense to require that path splittings of ϵ_ν are multiplicative.

For the second condition, we recall from Section 3.1 that for $\gamma \in PX$ and a smoothing function ϕ we have a retraction $\phi_\gamma : [0, 1] \longrightarrow PX$ with $\phi_\gamma(0) = \text{id}_{\gamma(0)}$ and $\phi_\gamma(1) = \gamma \circ \phi$. A 1-form $\kappa \in \Omega^1(PX)$ is called *contractible*, if

$$\int_{\phi_\gamma} \kappa = 0$$

for all $\gamma \in PX$ and some (and thus all) smoothing functions ϕ .

Before continuing, let us try to elucidate path splittings with the following example.

Example 4.5. For S a l -dimensional compact oriented smooth manifold, possibly with boundary, we denote by $\text{ev} : C^\infty(S, X) \times S \longrightarrow X$ the evaluation map, and let

$$\tau_S : \Omega^k(X) \longrightarrow \Omega^{k-l}(C^\infty(S, X)) : \rho \longmapsto \int_S \text{ev}^* \rho$$

be the usual transgression of differential forms to the mapping space. For $\rho \in \Omega^{k+1}(X)$ we set $\epsilon := \tau_{S^1}(\rho) \in \Omega^k(LX)$ and $\kappa := \tau_{[0,1]}(\rho)|_{PX} \in \Omega^k(PX)$. We claim:

- κ is a path splitting for ϵ . To see this, consider a path $\gamma : [0, 1] \longrightarrow PG^{[2]}$, with $\gamma = (\gamma_1, \gamma_2)$ and the associated homotopies $h_{\gamma_i} : [0, 1]^2 \longrightarrow G$; $h_{\gamma_i}(t, s) = \gamma_i(t)(s)$. Note that

$$\int_{\gamma_i} \kappa = \int_0^1 \kappa_{\gamma_i(t)}(\partial_t \gamma_i(t)) dt = \int_0^1 \int_0^1 \rho_{\gamma_i(t)(s)}(\partial_s \gamma_i(t)(s), \partial_t \gamma_i(t)(s)) ds dt = - \int_{h_{\gamma_i}} \rho.$$

We have

$$\int_{\gamma} \cup^* \epsilon = \int_0^1 \int_0^1 \rho_{(\cup \circ \gamma)(t)(s)}(\partial_s (\cup \circ \gamma)(t)(z), \partial_t (\cup \circ \gamma)(t)(s)) ds dt.$$

Splitting the integral over s in two parts (from 0 to $\frac{1}{2}$) and (from $\frac{1}{2}$ to 1), expressing it in terms of γ_1 and γ_2 , and reparameterizing, we get

$$\int_{\gamma} \cup^* \epsilon = \int_{h_{\gamma_1}} \rho - \int_{h_{\gamma_2}} \rho = \int_{\gamma} \text{pr}_2^* \kappa - \text{pr}_1^* \kappa.$$

But two 1-forms coincide if their integrals along all paths coincide; this shows $\cup^* \epsilon = \text{pr}_2^* \kappa - \text{pr}_1^* \kappa$.

- If X is a Lie group and ρ is multiplicative, then κ and ϵ are multiplicative, too. Indeed, τ_S is linear and natural with respect to smooth maps between smooth manifolds, and so commutes with the differential Δ .
- If $k = 1$, then κ is contractible: if $\gamma \in PX$, then we have

$$\int_{\phi_\gamma} \kappa = \int_{\phi_\gamma} \int_{[0,1]} \text{ev}^* \rho = \int_{[0,1]^2} (\text{ev} \circ (\phi_\gamma \times \text{id}))^* \rho = 0,$$

because $(\text{ev} \circ (\phi_\gamma \times \text{id}))(t, s) = \gamma(t\phi(s))$ is a rank one map.

One can show that the error 1-form $\epsilon_\nu \in \Omega^1(LG \times LG)$ of a fusive connection ν on \mathcal{L} admits a path splitting, and for compact groups G even a multiplicative path splitting. However, I do not know conditions that would guarantee the existence of a *contractible* path splitting. This constitutes our multiplicativity condition.

Definition 4.6. A multiplicative and fusive thin homotopy equivariant structure d is called multiplicative and fusive thin structure, if there exists a fusive superficial connection ν with $d^\nu = d$, whose error 1-form ϵ_ν admits a multiplicative and contractible path splitting.

Definition 4.7. Let G be a Lie group and $LG = C^\infty(S^1, G)$ be its loop group.

(a) A thin fusion extension of LG is a central extension

$$1 \longrightarrow \text{U}(1) \longrightarrow \mathcal{L} \longrightarrow LG \longrightarrow 1$$

together with a multiplicative fusion product and a multiplicative and fusive thin structure.

- (b) An isomorphism between thin fusion extensions is a smooth isomorphism between central extensions that is fusion-preserving and thin.

Thin fusion extensions form a category that we denote by $\mathcal{FusExt}^{th}(LG)$.

Before coming to examples, we shall investigate the following interesting feature of a thin fusion extension (Proposition 4.9 below). We recall that the Lie algebra of LG is $L\mathfrak{g} = C^\infty(S^1, \mathfrak{g})$ with all operations defined pointwise. We denote the Lie algebra of a central extension \mathcal{L} by \mathfrak{l} . It is a central extension

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathfrak{l} \xrightarrow{p_*} L\mathfrak{g} \longrightarrow 0$$

of Fréchet Lie algebras. We recall that a *splitting* is a continuous linear map $\delta : L\mathfrak{g} \longrightarrow \mathfrak{l}$ such that $p_* \circ \delta = \text{id}_{L\mathfrak{g}}$. Every connection ν on \mathcal{L} induces – via horizontal lift – a splitting δ_ν .

An element $\mathcal{X} \in L\mathfrak{g}$ is called *linear loop*, if there exist a smooth map $f : S^1 \longrightarrow \mathbb{R}$ and $X \in \mathfrak{g}$ such that $\mathcal{X}(z) = f(z)X$ for all $z \in S^1$. The linear loops span $L\mathfrak{g}$. Every linear loop \mathcal{X} can be represented – as a tangent vector – by a *thin* curve, namely by $\gamma_{\mathcal{X}} : \mathbb{R} \longrightarrow LG : t \mapsto \exp(t\mathcal{X})$, with $\gamma_{\mathcal{X}}(t)(z) = \exp(tf(z)X)$. Note that $t \mapsto (1, \gamma_{\mathcal{X}}(t))$ is a smooth curve in LG_{thin}^2 . Thus, a thin homotopy equivariant structure d on \mathcal{L} produces a smooth curve $d_{\mathcal{X}} : \mathbb{R} \longrightarrow \mathcal{L} : t \mapsto d_{1, \gamma_{\mathcal{X}}(t)}(1)$.

Lemma 4.8. *Suppose \mathcal{L} is a central extension with a thin homotopy equivariant structure d . Let ν be a thin connection on \mathcal{L} with $d^\nu = d$. Then, the splitting δ_ν is $\text{Diff}^+(S^1)$ -equivariant and satisfies*

$$\delta_\nu(\mathcal{X}) = \left. \frac{d}{dt} \right|_0 d_{\mathcal{X}}(t)$$

for all linear loops \mathcal{X} .

Proof. We calculate

$$\delta_\nu(\mathcal{X}) = \left. \frac{d}{dt} \right|_0 p_{\gamma_{\mathcal{X}}}^\nu(1, t) = \left. \frac{d}{dt} \right|_0 d_{1, \gamma_{\mathcal{X}}(t)}(1) = \left. \frac{d}{dt} \right|_0 d_{\mathcal{X}}(t). \quad (4.3)$$

Next we consider $\varphi \in \text{Diff}^+(S^1)$ and a linear loop \mathcal{X} . Then, $\exp(t\mathcal{X}) \circ \varphi = \exp(t(\mathcal{X} \circ \varphi))$ is thin homotopic to $\exp(t\mathcal{X})$, and we have

$$d_{\mathcal{X}}(t) \cdot \varphi = d_{1, \exp(t\mathcal{X})}(1) \cdot \varphi = d_{\exp(t\mathcal{X}), \exp(t(\mathcal{X} \circ \varphi))}(d_{1, \exp(t\mathcal{X})}(1)) = d_{1, \exp(t(\mathcal{X} \circ \varphi))}(1) = d_{\mathcal{X} \circ \varphi}(t).$$

Taking derivatives and using (4.3), we obtain $\delta_\nu(\mathcal{X}) \cdot \varphi = \delta_\nu(\mathcal{X} \circ \varphi)$. Since the $\text{Diff}^+(S^1)$ -actions on $L\mathfrak{g}$ and \mathfrak{l} are linear, and $L\mathfrak{g}$ is spanned by the linear loops, we conclude that δ_ν is equivariant. \square

In case of a thin fusion extension, we obtain:

Proposition 4.9. *Let \mathcal{L} be a thin fusion extension with thin structure d . Then, there exists a unique splitting $\delta : L\mathfrak{g} \longrightarrow \mathfrak{l}$ of the Lie algebra extension, such that*

$$\delta(\mathcal{X}) = \left. \frac{d}{dt} \right|_0 d_{\mathcal{X}}(t)$$

for all linear loops \mathcal{X} . Moreover, this splitting is $\text{Diff}^+(S^1)$ -equivariant.

Proof. Uniqueness follows because the linear loops span $L\mathfrak{g}$. Existence uses the existence of a thin connection ν on \mathcal{L} such that $d = d^\nu$. The corresponding splitting δ_ν has the claimed properties by Lemma 4.8. \square

In the remainder of this section we discuss three examples.

Example 4.10. The universal central extension \mathcal{L}_G of a compact, simple, connected, simply-connected Lie group G is a thin fusion extension. Since \mathcal{L}_G is universal, it follows that *every* central extension of LG is a thin fusion extension. In the model of Example 2.6 we obtain a connection ν by declaring its parallel transport along a path $\gamma : [0, 1] \rightarrow LG$ via

$$pt_\gamma : \mathcal{L}_G|_{\gamma(0)} \rightarrow \mathcal{L}_G|_{\gamma(1)} : (\phi_0, z_0) \mapsto (\phi_1, z_0 \cdot e^{2\pi i S_{\text{WZ}}(\Phi_\gamma)}), \quad (4.4)$$

where $\phi_1 : D^2 \rightarrow G$ is arbitrarily chosen such that $\partial\phi_1 = \gamma(1)$, and Φ_γ is obtained from γ , ϕ_0 and ϕ_1 exactly as described in the definition of the thin homotopy equivariant structure d on \mathcal{L}_G . In order to show that (4.4) defines a connection, it suffices to check (see [SW09, Theorem 5.4]):

- (a) It is compatible with the concatenation of paths: this is obvious.
- (b) It depends only on the thin homotopy class of the path, i.e. if γ and γ' are paths in LG with common initial loop and common end loop, and $h : [0, 1]^2 \rightarrow LG$ is a smooth map with $h(0, t) = \gamma(t)$ and $h(1, t) = \gamma'(t)$ (i.e. h is a fixed-ends-homotopy) and with the property that $\int h^* \omega = 0$ for all 2-forms $\omega \in \Omega^2(LG)$ (i.e. h is thin), then $pt_{\gamma_1} = pt_{\gamma_2}$. That this is the case can be seen by expressing the difference of the parallel transport maps (4.4) as the integral of $\omega := \tau_{S^1}(H)$ along h , which thus vanishes.
- (c) It depends smoothly on the path. This can be checked on smooth one-parameter family of paths, for which smoothness follows from the one of the integral of differential forms.

Thus, we have a connection ν on \mathcal{L}_G . It is straightforward to see that it is compatible with the fusion product λ . We have already seen in Example 2.6 that for a thin path γ the parallel transport pt_γ is independent of the choice of the thin path: this shows that ν is thin and induces d . It is also superficial: if two loops $\tau_1, \tau_2 \in LLG$ are homotopic via a homotopy h , then the difference between their holonomies is given by $\exp 2\pi i \int_h H$, where h is the homotopy. When h has rank two, the difference vanishes. The curvature of ν is $\tau_{S^1}(H)$, and the error 1-form is $\epsilon_\nu = \tau_{S^1}(\rho)$. Example 4.5 shows that ϵ_ν has a multiplicative and contractible path splitting.

Example 4.11. For any Lie group G , the central extension $\mathcal{L}_P = U(1) \times LG$ with the group structure defined from the holonomy η of a principal $U(1)$ -bundle P over $G \times G$, the trivial fusion product and the trivial thin structure (see Example 2.5), is a thin fusion extension. Indeed, an integrating connection ν can be obtained from any 2-form $\omega \in \Omega^2(G)$ by $\nu := \tau_{S^1}(\omega) \in \Omega^1(LG)$, e.g. $\omega = 0$ works. It is easy to see that this gives a fusive superficial connection, and that it induces the trivial thin structure, see [Wal12c, Proposition 3.1.8].

Example 4.12. We construct a central extension that cannot be equipped with the structure of a thin fusion extension. We work with $G = U(1)$, and consider $\mathcal{L} = U(1) \times LU(1)$ equipped with the group structure induced by the following 2-cocycle $\eta : LU(1) \times LU(1) \rightarrow U(1)$. If $\tau \in LU(1)$, we denote by $n \in \mathbb{Z}$ the winding number of τ , and find a smooth map $f : \mathbb{R} \rightarrow \mathbb{R}$ such that $f(t+1) = f(t) + n$ and $\tau = e^{2\pi i f}$. Note that f is determined by τ up to a shift by a constant $z \in \mathbb{Z}$. We define for $f : \mathbb{R} \rightarrow \mathbb{R}$ the average

$$\widehat{f} := \int_0^1 f(s) ds.$$

We define for $\alpha, \beta, \gamma \in \mathbb{R}$:

$$\eta(\tau_1, \tau_2) = \exp 2\pi i \left(\alpha \int_0^1 f_1(s) f_2'(s) ds + \beta (n_1 \widehat{f}_2 + n_2 \widehat{f}_1) + \gamma n_1 f_2(0) \right).$$

It is straightforward to check that the cocycle condition is satisfied for arbitrary values of α, β, γ . We remark that the 2-cocycle η is normalized, i.e. $\eta(1, 1) = 1$, for all parameters. We have to assure that η is well-defined under shifting f_k by integers z_k . We get

$$\begin{aligned} \int_0^1 (f_1(s) + z_1)(f_2 + z_2)'(s)ds &= \int_0^1 f_1(s)f_2'(s)ds + z_1n_2 \\ n_1\widehat{f_2 + z_2} + n_2\widehat{f_1 + z_1} &= n_1\widehat{f_2} + n_2\widehat{f_1} + n_1z_2 + n_2z_1 \\ n_1(f_2 + z_2)(0) &= n_1f_2(0) + n_1z_2. \end{aligned}$$

Note that all differences that arise are integers. We see two options to obtain well-definedness:

1. We choose the constants α, β, γ such that all differences cancel: $\beta = -\gamma$ and $\alpha = \gamma$. Then we have an \mathbb{R} -family of well-defined 2-cocycles. The corresponding central extensions are denoted by $\mathcal{L}_{\mathbb{R}}(\gamma)$.
2. We let $\alpha, \beta, \gamma \in \mathbb{Z}$ be arbitrary integers. Then, all differences vanish separately under exponentiation. This gives a \mathbb{Z}^3 -family of well-defined 2-cocycles. The corresponding central extensions are denoted by $\mathcal{L}_{\mathbb{Z}}(\alpha, \beta, \gamma)$.

We have coincidence $\mathcal{L}_{\mathbb{R}}(k) = \mathcal{L}_{\mathbb{Z}}(k, -k, k)$ for all $k \in \mathbb{Z}$. We observe that $\mathcal{L}_{\mathbb{Z}}(-1, 0, 1)$ is the extension \mathcal{L}_P of Examples 2.5 and 4.11, with P the Poincaré bundle over $T = \mathrm{U}(1) \times \mathrm{U}(1)$. Further, $\mathcal{L}_{\mathbb{Z}}(1, 1, -1)$ is the “basic central extension” of $LU(1)$ [PS86, Prop. 4.7.5].

One can show that for all τ_1, τ_2 , and all $\alpha, \beta, \gamma \in \mathbb{R}$ the following symmetry law holds:

$$\eta(\tau_1, \tau_2) = \exp 2\pi i \left(2\alpha \int_0^1 f_1(s)f_2'(s)ds - \alpha n_2 n_1 + (\gamma - \alpha)n_1 f_2(0) - (\gamma + \alpha)n_2 f_1(0) \right) \eta(\tau_2, \tau_1).$$

This shows in the first place that the extensions $\mathcal{L}_{\mathbb{Z}}(0, \beta, 0)$ are commutative. Let $\phi : [0, 1] \rightarrow [0, 1]$ be a smoothing map. Define smooth maps $f_1, f_2 : [0, 1] \rightarrow \mathbb{R}$ by

$$f_1(t) := \begin{cases} \phi(2t) & 0 \leq t \leq \frac{1}{2} \\ 0 & \frac{1}{2} \leq t \leq 1 \end{cases} \quad \text{and} \quad f_2(t) := \begin{cases} 1 & 0 \leq t \leq \frac{1}{2} \\ \phi(2t - 1) & \frac{1}{2} \leq t \leq 1 \end{cases}$$

and extend them periodically with shifts by $n_1 = n_2 = -1$. The corresponding loops τ_1 and τ_2 have disjoint support. We have $f_1 f_2' = 0$, $f_1(0) = f_2(0) = 1$, and hence

$$\eta(\tau_1, \tau_2) = \exp(2\pi i \alpha) \cdot \eta(\tau_2, \tau_1).$$

This shows that the central extensions $\mathcal{L}_{\mathbb{R}}(\gamma)$ with $\gamma \notin \mathbb{Z}$ do not satisfy the disjoint commutativity law of Theorem 3.3.1, and we conclude that they cannot be equipped with the structure of thin fusion extensions.

5 Transgression-regression machine

5.1 Multiplicative bundle gerbes

We use the theory of bundle gerbes and connections on those [Mur96, Ste00, CJM02, Wal07b]. We denote by $\mathcal{Grb}(X)$ and $\mathcal{Grb}^{\nabla}(X)$ the bicategories of bundle gerbes without and with connection over a smooth manifold X , respectively. Forgetting the connection is an essentially surjective, and in

general neither full nor faithful functor $\mathcal{G}rb^\nabla(X) \rightarrow \mathcal{G}rb(X)$. The 1-morphisms are called (connection-preserving) *isomorphisms*, and the 2-morphisms are called (connection-preserving) *transformations*. 2-forms $\rho \in \Omega^2(X)$ can be considered as connections on the trivial bundle gerbe \mathcal{I} ; as a bundle gerbe with connection it is denoted by \mathcal{I}_ρ .

Definition 5.1.1 ([CJM⁺05, Wal10]). *A multiplicative bundle gerbe with connection over a Lie group G is a bundle gerbe \mathcal{G} with connection over G , a multiplicative 2-form $\rho \in \Omega^2(G \times G)$, a connection-preserving isomorphism*

$$\mathcal{M} : \mathcal{G}_1 \otimes \mathcal{G}_2 \rightarrow \mathcal{G}_{12} \otimes \mathcal{I}_\rho$$

over $G \times G$, and a connection-preserving transformation

$$\begin{array}{ccc} \mathcal{G}_1 \otimes \mathcal{G}_2 \otimes \mathcal{G}_3 & \xrightarrow{\mathcal{M}_{1,2} \otimes \text{id}} & \mathcal{G}_{12} \otimes \mathcal{G}_3 \otimes \mathcal{I}_{\rho_{1,2}} \\ \text{id} \otimes \mathcal{M}_{2,3} \downarrow & \swarrow \alpha & \downarrow \mathcal{M}_{12,3} \otimes \text{id} \\ \mathcal{G}_1 \otimes \mathcal{G}_{23} \otimes \mathcal{I}_{\rho_{2,3}} & \xrightarrow{\mathcal{M}_{1,23} \otimes \text{id}} & \mathcal{G}_{123} \otimes \mathcal{I}_{\rho_\Delta} \end{array} \quad (5.1.1)$$

between isomorphisms over $G \times G \times G$, such that α satisfies a pentagon axiom over G^4 .

Here our index convention is so that e.g. the index $(\cdot)_{ij,k}$ stands for the pullback along the map $(g_i, g_j, g_k) \mapsto (g_i g_j, g_k)$. For instance, $\mathcal{G}_i = \text{pr}_i^* \mathcal{G}$ and \mathcal{G}_{12} is the pullback along the multiplication of G . Further, we have written $\rho_\Delta := \rho_{1,2} + \rho_{12,3} = \rho_{2,3} + \rho_{1,23}$, with the equality coming from the multiplicativity of ρ , see (4.2). The pentagon axiom for α can be found in [Wal10, Definition 1.3]. For later purpose, we need the following lemma.

Lemma 5.1.2. *Suppose G is connected. Let \mathcal{G} be a bundle gerbe with connection over G , $\rho \in \Omega^2(G \times G)$ be a multiplicative 2-form, and $\mathcal{M} : \mathcal{G}_1 \otimes \mathcal{G}_2 \rightarrow \mathcal{G}_{12} \otimes \mathcal{I}_\rho$ be a connection-preserving isomorphism, such that the set \mathcal{X} of connection-preserving transformations (5.1.1) is non-empty. Then, \mathcal{X} contains a unique element α that satisfies the pentagon axiom.*

Proof. We pick some $\alpha \in \mathcal{X}$. The pentagon axiom is an equality between two connection-preserving transformations over G^4 . Set of connection-preserving transformations between two fixed connection-preserving isomorphisms is a torsor over the group of locally constant $\text{U}(1)$ -valued maps. Thus, the pentagon axiom for α is satisfied up to a locally constant map $\epsilon : G^4 \rightarrow \text{U}(1)$. Since G is connected, ϵ is constant; $\epsilon \in \text{U}(1)$. We regard this constant as a locally constant map $\epsilon : G^3 \rightarrow \text{U}(1)$, and define a new element $\alpha' := \alpha \cdot \epsilon \in \mathcal{X}$. The pentagon axiom has five occurrences of α' : three on one side and two on the other. Thus, four occurrences of ϵ cancel, and the remaining one compensates the error caused by α ; hence, α' satisfies the pentagon axiom. Assume $\alpha, \alpha' \in \mathcal{X}$ satisfy the pentagon axiom. They differ by a locally constant map $\epsilon : G^3 \rightarrow \text{U}(1)$, i.e. a constant. In the pentagon axioms for α and α' , this leads to $\epsilon^3 = \epsilon^2$, i.e. $\epsilon = 1$. Therefore, $\alpha = \alpha'$. \square

If $(\mathcal{G}, \rho, \mathcal{M}, \alpha)$ and $(\mathcal{G}', \rho, \mathcal{M}', \alpha')$ are multiplicative bundle gerbes over G with connections (with the same 2-form ρ), a *1-morphism* is a connection-preserving isomorphism $\mathcal{A} : \mathcal{G} \rightarrow \mathcal{G}'$ together with

a connection-preserving transformation

$$\begin{array}{ccc}
\mathcal{G}_1 \otimes \mathcal{G}_2 & \xrightarrow{\mathcal{M}} & \mathcal{G}_{12} \otimes \mathcal{I}_\rho \\
\mathcal{A}_1 \otimes \mathcal{A}_2 \downarrow & \beta \swarrow \nearrow & \downarrow \mathcal{A}_{12} \otimes \text{id} \\
\mathcal{G}'_1 \otimes \mathcal{G}'_2 & \xrightarrow{\mathcal{M}'} & \mathcal{G}'_{12} \otimes \mathcal{I}_{\rho'}
\end{array} \tag{5.1.2}$$

over $G \times G$ that satisfies a compatibility condition with respect to α and α' over G^3 , see [Wal10, Definition 1.7].

Lemma 5.1.3. *Suppose $(\mathcal{G}, \rho, \mathcal{M}, \alpha)$ and $(\mathcal{G}', \rho, \mathcal{M}', \alpha')$ are multiplicative bundle gerbes with connection over a connected Lie group G , suppose $\mathcal{A} : \mathcal{G} \rightarrow \mathcal{G}'$ is a connection-preserving isomorphism, and suppose β is a connection-preserving transformation (5.1.2). Then, β is compatible with α and α' .*

Proof. We argue as in the proof of Lemma 5.1.2. The compatibility condition is an equality between two transformations over G^3 , with four occurrences of β . Thus, it is satisfied up to a locally constant map $\epsilon : G^3 \rightarrow \text{U}(1)$, i.e. a constant. The two pentagon axioms for α and α' over G^4 are related by $20 = 4 \cdot 5$ occurrences of β . As the pentagon axioms are satisfied, and the compatibility diagrams commute up to ϵ , we obtain that five occurrences of ϵ have to cancel. This requires $\epsilon = 1$. \square

If (\mathcal{A}, β) and (\mathcal{A}', β') are 1-morphisms between multiplicative bundle gerbes with connection, a *2-morphism* is a connection-preserving transformation $\varphi : \mathcal{A} \Rightarrow \mathcal{A}'$ such that the diagram

$$\begin{array}{ccc}
(\mathcal{A}_1 \otimes \mathcal{A}_2) \circ \mathcal{M} & \xRightarrow{\beta} & \mathcal{M}' \circ \mathcal{A}_{12} \\
(\varphi_1 \otimes \varphi_2) \circ \text{id} \downarrow & & \downarrow \text{id} \circ \varphi_{12} \\
(\mathcal{A}'_1 \otimes \mathcal{A}'_2) \circ \mathcal{M} & \xRightarrow{\beta'} & \mathcal{M}' \circ \mathcal{A}'_{12}
\end{array}$$

is commutative. With these definitions, multiplicative bundle gerbes with connection form a bicategory that we denote by $\text{MultGrb}^\nabla(G)$.

Multiplicative bundles gerbes *without* connections are defined analogously, without the 2-form ρ and without occurrences of trivial gerbes. We denote by $\text{MultGrb}(G)$ the bicategory of multiplicative bundle gerbes over G . We have the following classification result of [CJM⁺05]:

$$h_0 \text{MultGrb}(G) \cong H^4(BG, \mathbb{Z}), \tag{5.1.3}$$

where h_0 denotes taking the set of isomorphic objects. We denote by $\text{MultGrb}^\infty(G)$ the full sub-bicategory of $\text{MultGrb}(G)$ over those multiplicative bundle gerbes that *admit* connections. For compact Lie groups G , we have $\text{MultGrb}^\infty(G) = \text{MultGrb}(G)$ [Wal10, Proposition 2.8]. All bicategories of multiplicative bundle gerbes are symmetric monoidal, under the tensor product of bundle gerbes, and (5.1.3) is an isomorphism between groups.

Example 5.1.4. The trivial gerbe \mathcal{I}_ω for any $\omega \in \Omega^2(G)$ carries multiplicative structures parameterized by principal $\text{U}(1)$ -bundles P with connection over $G \times G$ such that

$$P_{1,2} \otimes P_{12,3} \cong P_{2,3} \otimes P_{1,23}$$

via a coherent connection-preserving isomorphism, see [Wal10, Example 1.4]. In this case, the 2-form is $\rho = \text{curv}(P) - \omega$. For $G = \text{U}(1)$, we have $H^4(\text{BU}(1), \mathbb{Z}) = H^4(K(\mathbb{Z}, 2), \mathbb{Z}) = \mathbb{Z}$. In [Wal10, Prop. 2.4] it is shown that there is an exact sequence

$$0 \longrightarrow H^3(\text{BU}(1), \text{U}(1)) \longrightarrow h_0 \text{MultGrb}^\nabla(\text{U}(1)) \longrightarrow \mathbb{Z}.$$

As $H^3(\text{BU}(1), \text{U}(1)) = H^3(K(\mathbb{Z}, 2), \text{U}(1)) = 0$, we see that $h_0 \text{MultGrb}^\nabla(\text{U}(1)) \cong h_0 \text{MultGrb}(\text{U}(1)) \cong \mathbb{Z}$. This \mathbb{Z} -family of multiplicative gerbes is obtained by taking $\omega = 0$ and P the Poincaré bundle over $T = \text{U}(1) \times \text{U}(1)$.

Example 5.1.5. Suppose G is compact, simple and simply-connected. There exists a (up to connection-preserving isomorphisms) unique bundle gerbe \mathcal{G}_{bas} with connection of curvature H , where $H \in \Omega^3(G)$ is the 3-form of Example 2.6; it is called the *basic gerbe*. One can show that \mathcal{G}_{bas} has a unique multiplicative structure [Wal10, Example 1.5], where $\rho \in \Omega^2(G \times G)$ is the 2-form of Example 2.6. There exist Lie-theoretical constructions of \mathcal{G}_{bas} [GR02, Mei02]. Constructions of the corresponding multiplicative structures are notoriously difficult; one option is described in [Wal12a, Section 7]. For (non-simply connected) compact simple Lie groups, all multiplicative gerbes with connection are tabulated, and can be constructed via descent from their simply-connected covers [GW09].

5.2 Transgressive central extensions

For every smooth manifold X , there is a transgression functor

$$\mathcal{T} : h_1 \text{Grb}^\nabla(X) \longrightarrow \text{Bun}(LX) \quad (5.2.1)$$

with target the category of Fréchet principal $\text{U}(1)$ -bundles over LX . The symbol h_1 stands for passing from a bicategory to a category by identifying 2-isomorphic isomorphisms.

Transgression for gerbes has been defined by Gawędzki in terms of cocycles for Deligne cohomology [Gaw88], and by Gawędzki-Reis for bundle gerbes [GR02]. Brylinski has defined transgression in terms of sheaves of categories [Bry93]. The functor (5.2.1) that we use here is an adaption of Brylinski's functor to bundle gerbes, and defined in [Wal10]. It is monoidal, and natural with respect to smooth maps $f : X \longrightarrow X'$ between smooth manifolds and the induced maps $Lf : LX \longrightarrow LX'$ between their loop spaces. Furthermore, if $\rho \in \Omega^2(X)$, \mathcal{I}_ρ is the trivial bundle gerbe with connection ρ , then its transgression $\mathcal{T}_{\mathcal{I}_\rho}$ has a canonical trivialization $t_\rho : \mathcal{T}_{\mathcal{I}_\rho} \longrightarrow \mathbf{I}$, where \mathbf{I} is the trivial $\text{U}(1)$ -bundle over LX .

Suppose $(\mathcal{G}, \rho, \mathcal{M}, \alpha)$ is a multiplicative bundle gerbe with connection over G . Applying the transgression functor to \mathcal{G} , we obtain a Fréchet principal $\text{U}(1)$ -bundle $\mathcal{L} := \mathcal{T}_{\mathcal{G}}$ over LG . Because transgression is functorial and monoidal, the transgression of the connection-preserving isomorphism \mathcal{M} together with the trivialization t_ρ give a bundle isomorphism

$$\mathcal{L}_1 \otimes \mathcal{L}_2 \xrightarrow{\mathcal{T}_{\mathcal{M}}} \mathcal{L}_{12} \otimes \mathcal{T}_{\mathcal{I}_\rho} \xrightarrow{\text{id} \otimes t_\rho} \mathcal{L}_{12}$$

over $LG \times LG$. It induces a binary operation on \mathcal{L} that covers the group structure of LG . The existence of the transformation α implies under transgression the associativity of that binary operation. This equips \mathcal{L} with the structure of a Fréchet Lie group [Wal10, Theorem 3.1.7], making up a central extension

$$1 \longrightarrow \text{U}(1) \longrightarrow \mathcal{L} \longrightarrow LG \longrightarrow 1.$$

Definition 5.2.1. A central extension \mathcal{L} of LG is called transgressive, if there exists a multiplicative bundle gerbe with connection over G whose transgression is isomorphic to \mathcal{L} as a central extension.

In [Walb] a category $\mathcal{FusBun}^{\nabla_{sf}}(LX)$ is considered with objects the Fréchet principal $U(1)$ -bundles over LX equipped with fusion products and fusive superficial connections, and morphisms the fusion-preserving, connection-preserving bundle morphisms. A construction in [Walb, Section 4.2] lifts the transgression functor (5.2.1) to this category:

$$\mathcal{T}^{\nabla} : \mathbf{h}_1 \mathcal{Grb}^{\nabla}(X) \longrightarrow \mathcal{FusBun}^{\nabla_{sf}}(LX). \quad (5.2.2)$$

In case of a multiplicative bundle gerbe \mathcal{G} with connection over G , this means in the first place that the underlying principal $U(1)$ -bundle of the central extension \mathcal{L} is equipped with a fusion product λ and with a fusive superficial connection ν .

Under the lifted transgression functor, the transgression $\mathcal{T}_{\mathcal{I}_{\rho}}^{\nabla}$ of the trivial bundle gerbe with connection $\rho \in \Omega^2(X)$ is equipped with a fusion product and a connection, which under the trivialization $t_{\rho} : \mathcal{T}_{\mathcal{I}_{\rho}}^{\nabla} \longrightarrow \mathbf{I}$ correspond to the trivial fusion product on \mathbf{I} and the connection 1-form $\epsilon_{\nu} := \tau_{S^1}(\rho) \in \Omega^1(LX)$ [Wal11, Lemma 3.6]. Thus, the group structure of \mathcal{L} is induced by the fusion-preserving, connection-preserving bundle morphism

$$\mathcal{L}_1 \otimes \mathcal{L}_2 \xrightarrow{\mathcal{T}_{\mathcal{M}}^{\nabla}} \mathcal{L}_{12} \otimes \mathcal{T}_{\mathcal{I}_{\rho}}^{\nabla} \xrightarrow{\text{id} \otimes t_{\rho}} \mathcal{L}_{12} \otimes \mathbf{I}_{\epsilon_{\nu}}.$$

This means (1) that $\epsilon_{\nu} \in \Omega^1(LG^2)$ is the error 1-form of ν . By Example 4.5, $\kappa := \tau_{[0,1]}(\rho)|_{P(G \times G)}$ is a multiplicative and contractible path splitting for ϵ_{ν} . It means (2) that the fusion product is multiplicative, see [Wala, Theorem 4.3.1].

Collecting all this data forces us to consider a category $\mathcal{FusExt}^{\nabla}(LG)$ with:

- Objects: Central extensions of LG equipped with a multiplicative fusion product, a fusive superficial connection, and a multiplicative, contractible path splitting of its error 1-form.
- Morphisms: Fusion-preserving, connection-preserving isomorphisms of central extensions, with the same error 1-form and the same path splitting on both sides.

The fact that transgression is a functor and monoidal implies that above procedure defines a monoidal functor

$$\mathcal{MT}^{\nabla} : \mathbf{h}_1 \mathcal{MultGrb}^{\nabla}(G) \longrightarrow \mathcal{FusExt}^{\nabla}(LG). \quad (5.2.3)$$

We can pass from superficial connections to thin structures in terms of an essentially surjective functor

$$th : \mathcal{FusExt}^{\nabla}(LG) \longrightarrow \mathcal{FusExt}^{th}(LG)$$

to the category of thin fusion extensions introduced in Section 4. Forgetting the fusion product and the thin structure gives another functor from $\mathcal{FusExt}^{th}(LG)$ to the category $\mathcal{Ext}(LG)$ of bare central extensions of LG . The composite

$$\mathbf{h}_1 \mathcal{MultGrb}^{\nabla}(G) \xrightarrow{\mathcal{MT}^{\nabla}} \mathcal{FusExt}^{\nabla}(LG) \xrightarrow{th} \mathcal{FusExt}^{th}(LG) \longrightarrow \mathcal{Ext}(LG) \quad (5.2.4)$$

is the procedure from the beginning of the present subsection: the transgression of a multiplicative bundle gerbe with connection to a central extension. Thus, we obtain the following result, constituting the first part of Theorem A.

Proposition 5.2.2. *A central extension is transgressive only if it can be equipped with the structure of a thin fusion extension.*

The functor (5.2.3) has a version when the connections on both sides are dropped, at the price that it only exists as map, not as a functor.

Proposition 5.2.3. *There exists a unique map $\mathcal{MT} : \mathrm{h}_0\mathrm{MultGrb}^\infty(G) \longrightarrow \mathrm{h}_0\mathrm{FusExt}^{th}(LG)$ such that the diagram*

$$\begin{array}{ccc} \mathrm{h}_0\mathrm{MultGrb}^\nabla(G) & \xrightarrow{\mathrm{h}_0\mathcal{MT}^\nabla} & \mathrm{h}_0\mathrm{FusExt}^\nabla(LG) \\ \downarrow & & \downarrow \mathrm{h}_0th \\ \mathrm{h}_0\mathrm{MultGrb}^\infty(G) & \xrightarrow{\mathcal{MT}} & \mathrm{h}_0\mathrm{FusExt}^{th}(LG) \end{array}$$

is commutative.

Proof. Uniqueness is clear as the vertical map on the left is surjective (by definition of $\mathrm{MultGrb}^\infty(G)$). For the existence, we prove that the thin fusion extensions one gets from different choices of connections on the same multiplicative bundle gerbe are isomorphic.

Let \mathcal{G} be a multiplicative bundle gerbe over G with two connections, say λ_1 and λ_2 , with corresponding thin fusion extensions \mathcal{L}_1 and \mathcal{L}_2 , respectively. In [NW13, Proposition 5.1.3] we have constructed an isomorphism $\varphi : \mathcal{L}_1 \longrightarrow \mathcal{L}_2$ between central extensions. It was defined as the composition $\varphi := (\mathrm{id} \otimes t_\beta) \circ \mathcal{T}_{\mathrm{id}_\epsilon}^\nabla$, where $\beta \in \Omega^2(G)$ and $\mathrm{id}_\epsilon : \mathcal{G}_{\lambda_1} \longrightarrow \mathcal{G}_{\lambda_2} \otimes \mathcal{I}_\beta$ is a connection-preserving isomorphism. The transgressed isomorphism $\mathcal{T}_{\mathrm{id}_\epsilon}^\nabla$ and the canonical trivialization $t_\beta : \mathcal{T}_{\mathcal{I}_\beta}^\nabla \longrightarrow \mathbf{I}_{\tau_{S^1}(\beta)}$ are fusion-preserving; this shows that φ is fusion-preserving. As a connection on the trivial bundle, the 1-form $\tau_{S^1}(\beta)$ induces the trivial thin structure [Wal12c, Proposition 3.2.3]; so that $th(t_\beta)$ is a thin bundle morphism. Hence, φ is an isomorphism of thin fusion extensions. \square

Example 5.2.4. We consider the transgression of the trivial gerbe \mathcal{I}_ω equipped with a multiplicative structure defined by a principal $\mathrm{U}(1)$ -bundle P with connection over $G \times G$ (see Example 5.1.4). The trivialization $t_\omega : \mathcal{T}_{\mathcal{I}_\omega}^\nabla \longrightarrow \mathrm{U}(1) \times LG$ induces an isomorphism between the transgressive thin fusion extension $\mathcal{T}_{\mathcal{I}_\omega}^\nabla$ and the thin fusion extension \mathcal{L}_P of Examples 2.5 and 4.11. In particular, \mathcal{L}_P is transgressive.

Example 5.2.5. Let G be a compact, simple, simply-connected Lie group, let \mathcal{G}_{bas} be the basic gerbe over G (Example 5.1.5), and let \mathcal{L}_G be the universal central extension of LG (Examples 2.6 and 4.10). There is an isomorphism

$$\varphi : \mathcal{L}_G \longrightarrow \mathcal{T}_{\mathcal{G}_{bas}}^\nabla$$

of central extensions that is fusion-preserving and thin, and so establishes an isomorphism between thin fusion extensions. In particular, the universal central extension \mathcal{L}_G is transgressive. The isomorphism φ is defined by

$$\varphi(\phi, z) = \partial\mathcal{T} \cdot z \cdot \exp 2\pi i \left(- \int_{D^2} \omega \right).$$

Here, $\mathcal{T}: \phi^* \mathcal{G}_{bas} \rightarrow \mathcal{I}_\omega$ is an arbitrarily chosen trivialization of $\phi^* \mathcal{G}_{bas}$ over D^2 and $\partial \mathcal{T}$ denotes its restriction to the boundary; the latter is a trivialization of $\partial \phi^* \mathcal{G}_{bas}$ over S^1 , constituting an element in $\mathcal{T}_{\mathcal{G}_{bas}}^\nabla$ over the loop $\partial \phi$. It is straightforward to see that φ is well-defined, $U(1)$ -equivariant, fusion-preserving, and a group homomorphism; see [Wala, Section 4.3] for details. In order to see that φ is thin, we consider the thin connection ν on \mathcal{L}_G that integrates the thin homotopy equivariant structure d (see Example 4.10), and show the stronger statement that φ is connection-preserving. Indeed, if $\gamma: [0, 1] \rightarrow LG$ is a path, and $\phi_0, \phi_1: D^2 \rightarrow G$ are smooth maps with $\partial \phi_0 = \gamma(0)$ and $\partial \phi_1 = \gamma(1)$, then $pt_\gamma^\nu(\phi_0, 1) = (\phi_1, e^{2\pi i S_{WZ}(\Phi_\gamma)})$. Let $\mathcal{T}_0, \mathcal{T}_1$ be trivializations of $\phi_0^* \mathcal{G}_{bas}$ and $\phi_1^* \mathcal{G}_{bas}$, respectively, and let $\tilde{\nu}$ denote the connection on $\mathcal{T}_{\mathcal{G}_{bas}}^\nabla$. Employing the definition of the transgression functor, see [Walb, Section 4.3], the parallel transport in $\mathcal{T}_{\mathcal{G}_{bas}}^\nabla$ is $pt_\gamma^{\tilde{\nu}}(\partial \mathcal{T}_0) = \partial \mathcal{T}_1 \cdot \mathcal{A}_{\mathcal{G}_{bas}}(h_\gamma, \mathcal{T}_0, \mathcal{T}_1)$, where the latter term is the surface holonomy of \mathcal{G}_{bas} with the trivializations as boundary conditions. In the present case of a 2-connected Lie group, it can be computed via the 3-form H and the two 2-forms ω_0, ω_1 of the trivializations $\mathcal{T}_0, \mathcal{T}_1$, namely as

$$\mathcal{A}_{\mathcal{G}_{bas}}(h_\gamma, \mathcal{T}_0, \mathcal{T}_1) = \exp 2\pi i \left(S_{WZ}(\Phi_\gamma) + \int_{D^2} \omega_0 - \int_{D^2} \omega_1 \right),$$

see [Wal07a, Proposition 3.1.4 (iii)]. Now we compute

$$\begin{aligned} pt_\gamma(\varphi(\phi_0, 1)) &= pt_\gamma(\partial \mathcal{T}_0) \cdot \exp 2\pi i \left(- \int_{D^2} \omega_0 \right) = \partial \mathcal{T}_1 \cdot \exp 2\pi i \left(S_{WZ}(\Phi_\gamma) - \int_{D^2} \omega_1 \right) \\ &= \varphi(\phi_1, e^{2\pi i S_{WZ}(\Phi_\gamma)}) = \varphi(pt_\gamma^\nu(\phi_0, 1)). \end{aligned}$$

This shows that φ commutes with the parallel transport along arbitrary paths; hence, φ is connection-preserving, in particular thin.

5.3 Regression and equivalence result

By the main result of [Walb], the lifted transgression functor (5.2.2) is an equivalence of categories, and has for fixed $x \in X$ a canonical inverse functor

$$\mathcal{R}_x^\nabla: \mathcal{FusBun}^{\nabla_{sf}}(LX) \rightarrow {}_{h_1}\mathcal{Grb}^\nabla(X), \quad (5.3.1)$$

called *regression*. We need the following result about the regression of trivial bundles, which is explained in Appendix A. If $\epsilon \in \Omega^1(LX)$ is a superficial connection on the trivial bundle over LX , and fusive with respect to the trivial fusion product, then every path splitting $\kappa \in \Omega^1(PX)$ defines a connection-preserving isomorphism $\mathcal{T}_\kappa: \mathcal{R}^\nabla(\mathbf{I}_\epsilon) \rightarrow \mathcal{I}_{\rho_\kappa}$ between the regression of \mathbf{I}_ϵ and the trivial bundle gerbe \mathcal{I} over X equipped with a connection 2-form $\rho_\kappa \in \Omega^2(X)$ defined from κ .

If X is a group, we always choose $x = 1$ and omit the index. We use the functor $\mathcal{R}^\nabla = \mathcal{R}_1^\nabla$ in order to construct a regression functor

$$\mathcal{MR}^\nabla: \mathcal{FusExt}^\nabla(LG) \rightarrow {}_{h_1}\mathcal{MultGrb}^\nabla(G) \quad (5.3.2)$$

defined on the category $\mathcal{FusExt}^\nabla(LG)$ introduced in the previous section. Suppose \mathcal{L} is a central extension of LG equipped with a multiplicative fusion product λ , a fusive superficial connection, and a multiplicative path splitting κ of its error 1-form ϵ (for the definition of \mathcal{MR}^∇ we do not need that κ is contractible). The group structure defines over $LG \times LG$ a connection-preserving, fusion-preserving bundle isomorphism

$$\mu: \mathcal{L}_1 \otimes \mathcal{L}_2 \rightarrow \mathcal{L}_{12} \otimes \mathbf{I}_\epsilon,$$

and the associativity of the group structure implies a commutative diagram over LG^3 .

We let $\mathcal{G} := \mathcal{R}^\nabla(\mathcal{L})$ be the regressed bundle gerbe over G with connection. Over $G \times G$ we consider the connection-preserving isomorphism \mathcal{M} defined as

$$\mathcal{G}_1 \otimes \mathcal{G}_2 \xrightarrow{\mathcal{R}^\nabla(\mu)} \mathcal{G}_{12} \otimes \mathcal{R}^\nabla(\mathbf{I}_\epsilon) \xrightarrow{\text{id} \otimes \mathcal{T}_\kappa} \mathcal{G}_{12} \otimes \mathcal{I}_{\rho_\kappa}. \quad (5.3.3)$$

The various pullbacks of \mathcal{M} to $G \times G \times G$ constitute the outer arrows of the following diagram in the category $\text{h}_1\text{Grb}^\nabla(G \times G \times G)$:

$$\begin{array}{ccccc}
\mathcal{G}_1 \otimes \mathcal{G}_2 \otimes \mathcal{G}_3 & \xrightarrow{\mathcal{M}_{1,2} \otimes \text{id}} & \mathcal{G}_{12} \otimes \mathcal{G}_3 \otimes \mathcal{I}_{\rho_{\kappa,1,2}} & & \\
\downarrow \text{id} \otimes \mathcal{R}^\nabla(\mu)_{2,3} & \searrow \mathcal{R}^\nabla(\mu)_{1,2} \otimes \text{id} & \searrow \text{id} \otimes \text{id} \otimes \mathcal{T}_{\kappa_{1,2}} & & \downarrow \mathcal{R}^\nabla(\mu)_{12,3} \otimes \text{id} \\
& \mathcal{G}_{12} \otimes \mathcal{G}_3 \otimes \mathcal{R}^\nabla(\mathbf{I}_{\epsilon_{1,2}}) & & & \\
& \downarrow \mathcal{R}^\nabla(\mu)_{12,3} \otimes \text{id} & \searrow \mathcal{R}^\nabla(\mu)_{12,3} \otimes \mathcal{T}_{\kappa_{1,2}} & & \\
& \mathcal{G}_{123} \otimes \mathcal{R}^\nabla(\mathbf{I}_{\epsilon_\Delta}) & \xrightarrow{\text{id} \otimes \mathcal{T}_{\kappa_{1,2}}} & \mathcal{G}_{123} \otimes \mathcal{R}^\nabla(\mathbf{I}_{\epsilon_{12,3}}) \otimes \mathcal{I}_{\rho_{\kappa,1,2}} & \\
& \downarrow \text{id} \otimes \mathcal{T}_{\kappa_{2,3}} & & \downarrow \text{id} \otimes \mathcal{T}_{\kappa_{12,3}} \otimes \text{id} & \\
& \mathcal{G}_{123} \otimes \mathcal{R}^\nabla(\mathbf{I}_{\epsilon_{1,23}}) \otimes \mathcal{I}_{\rho_{\kappa,2,3}} & & & \\
& \downarrow \mathcal{R}^\nabla(\mu)_{1,23} \otimes \text{id} & \searrow \text{id} \otimes \mathcal{T}_{\kappa_{1,23}} \otimes \text{id} & & \\
\mathcal{G}_1 \otimes \mathcal{G}_{23} \otimes \mathcal{I}_{\rho_{\kappa,2,3}} & \xrightarrow{\mathcal{M}_{1,23} \otimes \text{id}} & \mathcal{G}_{123} \otimes \mathcal{I}_{\rho_{\kappa,\Delta}} & & \\
\downarrow \text{id} \otimes \mathcal{M}_{2,3} & \searrow \mathcal{R}^\nabla(\mu)_{1,23} \otimes \text{id} & \searrow \text{id} \otimes \mathcal{T}_{\kappa_{1,23}} \otimes \text{id} & & \downarrow \mathcal{M}_{12,3} \otimes \text{id} \\
& \mathcal{G}_1 \otimes \mathcal{G}_{23} \otimes \mathcal{R}^\nabla(\mathbf{I}_{\epsilon_{2,3}}) & \xrightarrow{\mathcal{R}^\nabla(\mu)_{1,23} \otimes \text{id}} & \mathcal{G}_{123} \otimes \mathcal{R}^\nabla(\mathbf{I}_{\epsilon_\Delta}) & \\
& \downarrow \text{id} \otimes \text{id} \otimes \mathcal{T}_{\kappa_{2,3}} & \searrow \mathcal{R}^\nabla(\mu)_{1,23} \otimes \mathcal{T}_{\kappa_{2,3}} & & \\
& \mathcal{G}_1 \otimes \mathcal{G}_{23} \otimes \mathcal{I}_{\rho_{\kappa,2,3}} & & &
\end{array}$$

All triangular subdiagrams commute obviously. There remain two four-sided subdiagrams whose commutativity is to check. The one that touches the upper left corner commutes due to the commutative diagram for μ over LG^3 and the fact that \mathcal{R}^∇ is a functor. The one that touches the lower right corner commutes due to the multiplicativity of κ and the additivity of the trivialization \mathcal{T}_κ shown in Lemma A.2.

The commutativity of the diagram in $\text{h}_1\text{Grb}^\nabla(G^3)$ implies the existence of a connection-preserving transformation α that fills the diagram in $\text{Grb}^\nabla(G^3)$. Thus, by Lemma 5.1.2 there is a unique connection-preserving transformation α making $(\mathcal{G}, \rho_\kappa, \mathcal{M}, \alpha)$ a multiplicative bundle gerbe with connection; this defines the functor \mathcal{MR}^∇ on the level of objects. On the level of morphisms, one similarly composes a commutative diagram from an isomorphism in $\text{FusExt}^\nabla(LG)$, and then Lemma 5.1.3 implies that it yields a morphism between multiplicative bundle gerbes with connection.

Theorem 5.3.1. *For a connected Lie group G , the two functors \mathcal{MT}^∇ and \mathcal{MR}^∇ form an equivalence of categories:*

$$\text{h}_1\text{MultGrb}^\nabla(G) \cong \text{FusExt}^\nabla(LG).$$

Proof. We consider first the composite $\mathcal{MT}^\nabla \circ \mathcal{MR}^\nabla$. Let \mathcal{L} be a central extension of LG equipped with a fusion product λ , a fusive superficial connection, and a multiplicative, contractible path splitting κ of its error 1-form ϵ . We denote by $(\mathcal{G}, \rho_\kappa, \mathcal{M}, \alpha)$ the regressed multiplicative bundle gerbe over G . Let $\varphi : \mathcal{T}^\nabla \circ \mathcal{R}^\nabla \rightarrow \text{id}_{\text{FusBun}^\nabla(LG)}$ be the natural transformation that establishes one half of the

fact that \mathcal{T}^∇ and \mathcal{R}^∇ form an equivalence of categories. Thus, we have a connection-preserving, fusion-preserving isomorphism $\varphi_{\mathcal{L}} : \mathcal{T}_{\mathcal{G}}^\nabla \longrightarrow \mathcal{L}$ over LG . The diagram

$$\begin{array}{ccccc}
\text{pr}_1^* \mathcal{T}_{\mathcal{G}}^\nabla \otimes \text{pr}_2^* \mathcal{T}_{\mathcal{G}}^\nabla & \xrightarrow{\mathcal{T}_{\mathcal{M}}^\nabla} & m^* \mathcal{T}_{\mathcal{G}}^\nabla \otimes \mathcal{T}_{\mathcal{I}_{\rho\kappa}}^\nabla & \xrightarrow{\text{id} \otimes t_{\rho\kappa}} & m^* \mathcal{T}_{\mathcal{G}}^\nabla \otimes \mathbf{I}_\epsilon \\
\downarrow \text{pr}_1^* \varphi_{\mathcal{L}} \otimes \text{pr}_2^* \varphi_{\mathcal{L}} & \searrow \mathcal{T}_{\mathcal{R}^\nabla(\mu)} & \nearrow \text{id} \otimes \mathcal{T}_{\mathcal{I}_{\rho\kappa}}^\nabla & & \downarrow m^* \varphi_{\mathcal{L}} \otimes \text{id} \\
& m^* \mathcal{T}_{\mathcal{G}}^\nabla \otimes \mathcal{T}_{\mathcal{R}^\nabla(\mathbf{I}_\epsilon)}^\nabla & & & \\
& \downarrow m^* \varphi_{\mathcal{L}} \otimes \varphi_{\mathbf{I}_\epsilon} & & & \\
\text{pr}_1^* \mathcal{L} \otimes \text{pr}_2^* \mathcal{L} & \xrightarrow{\mu} & m^* \mathcal{L} \otimes \mathbf{I}_\epsilon & \xlongequal{\quad} & m^* \mathcal{L} \otimes \mathbf{I}_\epsilon
\end{array}$$

of bundle isomorphisms over $LG \times LG$ is commutative: the triangular diagram is the definition of the isomorphism \mathcal{M} , the left part commutes because φ is natural, and the right part commutes due to Proposition A.4 (this uses that κ is contractible). The bottom line is the multiplication of \mathcal{L} , and the top line is by definition the group structure on $\mathcal{T}_{\mathcal{G}}$. Hence, $\varphi_{\mathcal{L}}$ is connection-preserving, fusion-preserving and a group homomorphism.

Now we look at $\mathcal{MR}^\nabla \circ \mathcal{MT}^\nabla$. Let $(\mathcal{G}, \rho, \mathcal{M}, \alpha)$ be a multiplicative bundle gerbe with connection over G . Let \mathcal{L} be its transgression, with fusion product λ , multiplication isomorphism μ , error 1-form $\epsilon = \tau_{S^1}(\rho)$ and path splitting $\kappa = \tau_{[0,1]}(\rho)|_{P(G \times G)}$. Let $\mathcal{A} : \mathcal{R}^\nabla \circ \mathcal{T}^\nabla \longrightarrow \text{id}_{\text{Grb}^\nabla(G)}$ be the natural transformation that establishes the second half of the fact that \mathcal{T}^∇ and \mathcal{R}^∇ form an equivalence; thus, $\mathcal{A}_{\mathcal{G}} : \mathcal{R}^\nabla(\mathcal{L}) \longrightarrow \mathcal{G}$ is a connection-preserving isomorphism. The diagram

$$\begin{array}{ccccc}
\mathcal{R}^\nabla(\mathcal{L})_1 \otimes \mathcal{R}^\nabla(\mathcal{L})_2 & \xrightarrow{\mathcal{R}^\nabla(\mu)} & \mathcal{R}^\nabla(\mathcal{L})_{12} \otimes \mathcal{R}^\nabla(\mathbf{I}_\epsilon) & \xrightarrow{\text{id} \otimes \mathcal{T}_\kappa} & \mathcal{R}^\nabla(\mathcal{L})_{12} \otimes \mathcal{I}_{\rho\kappa} \\
\downarrow \text{pr}_1^* \mathcal{A}_{\mathcal{G}} \otimes \text{pr}_2^* \mathcal{A}_{\mathcal{G}} & \searrow \mathcal{R}^\nabla(\mathcal{T}_{\mathcal{M}}^\nabla) & \nearrow \text{id} \otimes \mathcal{R}^\nabla(t_\rho) & & \downarrow m^* \mathcal{A}_{\mathcal{G}} \otimes \text{id} \\
& \mathcal{R}^\nabla(\mathcal{L})_{12} \otimes \mathcal{R}^\nabla(\mathcal{T}_{\mathcal{I}_{\rho}}^\nabla) & & & \\
& \downarrow m^* \mathcal{A}_{\mathcal{G}} \otimes \mathcal{A}_{\mathcal{I}_{\rho}} & & & \\
\mathcal{G}_1 \otimes \mathcal{G}_2 & \xrightarrow{\quad} & \mathcal{G}_{12} \otimes \mathcal{I}_\rho & \xlongequal{\quad} & \mathcal{G}_{12} \otimes \mathcal{I}_\rho
\end{array}$$

in the category $\text{h}_1\text{Grb}^\nabla(G \times G)$ is commutative: the triangular diagram is the definition of the multiplication μ , the left part is the naturality of \mathcal{A} , and the right part is Proposition A.3. This means that there exists a connection-preserving transformation β that fills the diagram. Now we are in the situation of Lemma 5.1.3, saying that β satisfies the compatibility condition with respect to α and α' , and it becomes an isomorphism between multiplicative bundle gerbes with connection over G . \square

In the commutative diagram of Proposition 5.2.3 the maps $\text{h}_0 \mathcal{MT}^\nabla$ and $\text{h}_0 \text{th}$ are surjective (by Theorem 5.3.1 and by definition, respectively), hence the map

$$\mathcal{MT} : \text{h}_0 \text{MultGrb}^\infty(G) \longrightarrow \text{h}_0 \text{FusExt}^{th}(LG)$$

is surjective. Thus, we have the next part of Theorem A:

Corollary 5.3.2. *If G is connected, then every thin fusion extension of LG is transgressive.*

In order to complete the proof of Theorem A, we have to show that the map \mathcal{MT} is injective and so establishes a bijection.

Proposition 5.3.3. *There exists a unique map $\mathcal{MR} : \mathrm{h}_0\mathcal{FusExt}^{th}(LG) \longrightarrow \mathrm{h}_0\mathcal{MultGrb}^\infty(G)$ such that the diagram*

$$\begin{array}{ccc} \mathrm{h}_0\mathcal{FusExt}^\nabla(LG) & \xrightarrow{\mathrm{h}_0\mathcal{MR}^\nabla} & \mathrm{h}_0\mathcal{MultGrb}^\nabla(G) \\ \mathrm{h}_0th \downarrow & & \downarrow \\ \mathrm{h}_0\mathcal{FusExt}^{th}(LG) & \xrightarrow{\mathcal{MR}} & \mathrm{h}_0\mathcal{MultGrb}^\infty(G) \end{array}$$

is commutative.

Proof. Uniqueness of the map follows since h_0th is surjective. In order to define the map \mathcal{MR} , we infer that the regression functor \mathcal{R}^∇ of (5.3.1) covers a functor $\mathcal{R}_x : \mathcal{FusBun}(LX) \longrightarrow \mathrm{h}_1\mathcal{Grb}(X)$ on the level without connections [Walb, Section 5.1]. Using only the group structure and the multiplicativity of the fusion product, the bundle gerbe $\mathcal{R}(\mathcal{L})$ can be equipped with a so-called strictly multiplicative structure, which in turns induces a multiplicative structure, see Sections 2 and 5 of [Wal12a]. It remains to prove that the diagram is commutative. We assume that we have a fusive superficial connection ν on \mathcal{L} together with a multiplicative path splitting κ .

By construction, $\mathcal{MR}^\nabla(\mathcal{L})$ and $\mathcal{MR}(th(\mathcal{L}))$ have the same underlying bundle gerbe $\mathcal{G} = \mathcal{R}(\mathcal{L})$, and the same underlying isomorphism $\mathcal{M} = \mathcal{R}(\mu)$. We show that the associators coincide; in order to do so, we prove that the isomorphism \mathcal{M} and the associator α obtained from the strictly multiplicative structure are connection-preserving; thus, by Lemma 5.1.2 α equals the associator of $\mathcal{MR}^\nabla(\mathcal{L})$.

The isomorphism \mathcal{M} is induced from the map $r : Y_{1,2} \longrightarrow Y_{12}$ between the surjective submersions $Y_{1,2} := P_1G \times P_1G$ of $\mathcal{G}_1 \otimes \mathcal{G}_2$ and $Y_{12} := G \times P_1G$ of \mathcal{G}_{12} , and a lift $R : P_{1,2} \longrightarrow P_{12}$ of the map $r \times r$ to the total spaces of the principal $U(1)$ -bundles of these gerbes. Explicitly, $r(\gamma_1, \gamma_2) := (\gamma_1(1), \gamma_1\gamma_2)$, the bundles are $P_{1,2}|_{(\gamma_1, \gamma_2), (\gamma'_1, \gamma'_2)} = \mathcal{L}_{\gamma_1 \cup \gamma'_1} \otimes \mathcal{L}_{\gamma_2 \cup \gamma'_2}$ and $P_{12}|_{(g, \gamma), (g', \gamma')} = \mathcal{L}_{\gamma \cup \gamma'}$, and R is multiplication. An isomorphism induced from maps (r, R) is connection-preserving with respect to the induced connections $\nu_{1,2}$ on $P_{1,2}$ and ν_{12} on P_{12} , if there exists a 1-form $\kappa \in \Omega^1(Y_{1,2})$ such that $R^*\nu_{12} + \mathrm{pr}_2^*\kappa - \mathrm{pr}_1^*\kappa = \nu_{1,2}$ over $Y_{1,2} \times_G Y_{1,2}$. This is exactly the property of the path splitting κ ; hence \mathcal{M} is connection-preserving. The condition that κ is multiplicative then implies that α is connection-preserving. \square

Now, we have two maps \mathcal{MT} and \mathcal{MR} on the bottom of the commutative diagrams of Propositions 5.2.3 and 5.3.3, covered along surjective maps by maps $\mathrm{h}_0\mathcal{MT}^\nabla$ and $\mathrm{h}_0\mathcal{MR}^\nabla$ that are inverses of each other according to Theorem 5.3.1. This suffices to show the last part of Theorem A:

Corollary 5.3.4. *The maps \mathcal{MT} and \mathcal{MR} are inverses of each other, and establish a bijection*

$$\mathrm{h}_0\mathcal{MultGrb}^\infty(G) \cong \mathrm{h}_0\mathcal{FusExt}^{th}(LG).$$

6 Segal-Witten reciprocity

Let G be a Lie group and \mathcal{L} be a central extension of LG . Let Σ be a compact oriented surface with boundary components $b_1, \dots, b_k \subseteq \partial\Sigma$ parameterized by orientation-preserving diffeomorphisms

$f_i : S^1 \longrightarrow b_i$. We have induced maps $r_i : C^\infty(\Sigma, G) \longrightarrow LG$ defined by $r_i(\phi) := \phi \circ f_i$. We let \mathcal{L}_Σ denote the Baer sum of the central extensions $r_i^* \mathcal{L}$ of $C^\infty(\Sigma, G)$,

$$\mathcal{L}_\Sigma := r_1^* \mathcal{L} \otimes \dots \otimes r_k^* \mathcal{L}.$$

If Σ is obtained from two surfaces Σ_1 and Σ_2 by gluing along some boundary components, and $\rho_i : C^\infty(\Sigma, G) \longrightarrow C^\infty(\Sigma_i, G)$ are the restriction maps, then we have an isomorphism

$$\rho : \mathcal{L}_\Sigma \longrightarrow \rho_1^* \mathcal{L}_{\Sigma_1} \otimes \rho_2^* \mathcal{L}_{\Sigma_2}.$$

Definition 6.1. A central extension \mathcal{L} of LG has the smooth reciprocity property, if there exists a family $\{s_\Sigma\}$ of splittings s_Σ of \mathcal{L}_Σ for every compact oriented surface Σ , satisfying the gluing law

$$\rho \circ s_\Sigma = \rho_1^* s_{\Sigma_1} \otimes \rho_2^* s_{\Sigma_2},$$

whenever Σ is obtained from two surfaces Σ_1 and Σ_2 by gluing along some boundary components.

By splitting we mean a smooth map $s_\Sigma : C^\infty(\Sigma, G) \longrightarrow \mathcal{L}_\Sigma$ such that $p \circ s_\Sigma = \text{id}_{C^\infty(\Sigma, G)}$, where p is the projection $p : \mathcal{L}_\Sigma \longrightarrow C^\infty(\Sigma, G)$.

Remark 6.2.

- (i) Above definition is derived from a definition due to Brylinski and McLaughlin [BM94] and a gluing law from [BM96]; in these references the definition is attributed to Segal [Seg04].
- (ii) There is a *complex* version of the reciprocity property, where G is a complex Lie group, Σ is a Riemann surface, and $C^\infty(\Sigma, G)$ is replaced by $\text{Hol}(\Sigma, G)$, the holomorphic maps from Σ to G [BM96].
- (iii) The smooth reciprocity property is a property of the underlying principal $U(1)$ -bundle of \mathcal{L} , i.e. the group structure is neglected. For the complex reciprocity property one additionally assumes that the sections s_Σ are group homomorphisms.

The Segal-Witten reciprocity law states that every transgressive central extension of the loop group of a complex Lie group has the complex reciprocity property. The following result is a weaker version adapted to the *smooth* reciprocity property.

Theorem 6.3. Every transgressive central extension of the loop group of any Lie group has the smooth reciprocity property.

Proof. Let $(\mathcal{G}, \rho, \mathcal{M}, \alpha)$ be a multiplicative bundle gerbe with connection over G , and let \mathcal{L} be the corresponding central extension. Suppose Σ is a compact oriented surface, $\phi : \Sigma \longrightarrow G$ is a smooth map, and \mathcal{T}_i is a trivialization of $\phi^* \mathcal{G}|_{b_i}$ for every boundary component b_i . The *surface holonomy* $\mathcal{A}_\Sigma(\phi, \mathcal{T}_1, \dots, \mathcal{T}_k) \in U(1)$ of \mathcal{G} with boundary conditions $\mathcal{T}_1, \dots, \mathcal{T}_k$ is defined in the following way. Choose a trivialization $\mathcal{S} : \phi^* \mathcal{G} \longrightarrow \mathcal{I}_\omega$. For each boundary component, we have two trivializations that differ by a $U(1)$ -bundle T_i with connection over b_i , i.e. $\mathcal{S}|_{b_i} \cong \mathcal{T}_i \otimes T_i$. Then,

$$\mathcal{A}_\Sigma(\phi, \mathcal{T}_1, \dots, \mathcal{T}_k) := \exp \left(\int_\Sigma \omega \right) \cdot \prod_{i=1}^k \text{Hol}_{T_i}(b_i)^{-1}.$$

With boundary parameterizations $f_i : S^1 \rightarrow b_i$ we have $f_i^* \mathcal{T}_i \in (r_i^* \mathcal{L})_\phi$ and so $(f_1^* \mathcal{T}_1, \dots, f_k^* \mathcal{T}_k) \in \mathcal{L}_\Sigma|_\phi$. Now, a splitting $s_\Sigma : C^\infty(\Sigma, G) \rightarrow \mathcal{L}_\Sigma$ is defined by

$$s_\Sigma(\phi) = (f_1^* \mathcal{T}_1, \dots, f_k^* \mathcal{T}_k) \cdot \mathcal{A}_\Sigma(\phi, \mathcal{T}_1, \dots, \mathcal{T}_k).$$

This splitting satisfies the gluing property since surface holonomy has a more general gluing property, see [CJM02, Proposition 3.1] and [Walb, Lemma 3.3.3 (c)]. \square

Remark 6.4.

- (i) Theorem 6.3 is proved in [Bry93, Theorem 6.2.1] and [BM94, Theorem 5.9] for simply-connected Lie groups, and in the latter reference it is claimed that it generalizes to arbitrary Lie groups by a (left out) computation in simplicial Deligne cohomology.
- (ii) By Theorem A the transgressive central extensions are precisely the thin fusion extension. For a given thin fusion extension \mathcal{L} the splitting s_Σ can be constructed directly from a choice of a fusive superficial connection ν that integrates the given thin structure. Indeed, the surface holonomy $\mathcal{A}_\Sigma(\mathcal{T}_1, \dots, \mathcal{T}_k)$ can be defined directly from ν and the fusion product, see [Walb, Section 5.3].

It is an interesting question whether or not the sections s_Σ can be chosen multiplicative, i.e. to be group homomorphisms. In [BM94] it is claimed that this is possible for transgressive central extensions of loop groups of arbitrary Lie groups. In [Bry] on pages 2 and 21 Brylinski withdraws that statement, and claims that only the sections for the *complex* reciprocity property can be chosen multiplicative. Based on this claim, it is proved in [BM96] that the complex reciprocity property characterizes transgressive central extensions of the loop group of a connected semisimple complex Lie group.

In the present paper we make no claims about complex Lie groups. In the following we only show via examples that there exist transgressive central extensions for which the sections s_Σ cannot be chosen multiplicative (Example 6.6), and that there exist central extensions that have the smooth reciprocity property but are not transgressive (Example 6.7). For preparation, we need the following.

Proposition 6.5. *The splitting s_Σ constructed in Theorem 6.3 satisfies*

$$s_\Sigma(\phi_1 \phi_2) = s_\Sigma(\phi_1) \cdot s_\Sigma(\phi_2) \cdot \exp 2\pi i \left(- \int_\Sigma (\phi_1, \phi_2)^* \rho \right),$$

where $\rho \in \Omega^2(G \times G)$ is the 2-form of the multiplicative gerbe.

Proof. Suppose $\phi_1, \phi_2 : \Sigma \rightarrow G$. We may have chosen trivializations $\mathcal{S}_j : \phi_j^* \mathcal{G} \rightarrow \mathcal{I}_{\omega_j}$. Then, we get another trivialization \mathcal{S} defined by

$$(\phi_1 \phi_2)^* \mathcal{G} \xrightarrow{(\phi_1, \phi_2)^* \mathcal{M}^{-1}} \phi_1^* \mathcal{G} \otimes \phi_2^* \mathcal{G} \otimes \mathcal{I}_{-(\phi_1, \phi_2)^* \rho} \xrightarrow{\mathcal{S}_1 \otimes \mathcal{S}_2} \mathcal{I}_{\omega_1 + \omega_2 - (\phi_1, \phi_2)^* \rho}.$$

We further have trivializations $\mathcal{T}_{ij} : \phi_j^* \mathcal{G}|_{b_i} \rightarrow \mathcal{I}_0$ for $j = 1, 2$ and $i = 1, \dots, k$ and the difference bundles $\mathcal{S}_j|_{b_i} \cong \mathcal{T}_{ij} \otimes T_{ij}$. We consider the trivialization \mathcal{T}_i defined by

$$(\phi_1 \phi_2)^* \mathcal{G}|_{b_i} \xrightarrow{(\phi_1, \phi_2)^* \mathcal{M}^{-1}} \phi_1^* \mathcal{G}|_{b_i} \otimes \phi_2^* \mathcal{G}|_{b_i} \xrightarrow{\mathcal{T}_{i1} \otimes \mathcal{T}_{i2}} \mathcal{I}_0,$$

and obtain

$$\mathcal{S}|_{b_i} = (\mathcal{S}_1|_{b_i} \otimes \mathcal{S}_2|_{b_i}) \circ (\phi_1, \phi_2)^* \mathcal{M}^{-1}|_{b_i} \cong (\mathcal{T}_{i1} \otimes \mathcal{T}_{i2}) \circ (\phi_1, \phi_2)^* \mathcal{M}^{-1}|_{b_i} \otimes T_{i1} \otimes T_{i2} = \mathcal{T}_i \otimes T_{i1} \otimes T_{i2}.$$

Thus,

$$\begin{aligned}\mathcal{A}_\Sigma(\phi_1\phi_2, \mathcal{T}_1, \dots, \mathcal{T}_k) &= \exp 2\pi i \left(\int_\Sigma \omega_1 + \omega_2 - (\phi_1, \phi_2)^* \rho \right) \cdot \prod_{i=1}^k \text{Hol}_{T_{i1} \otimes T_{i2}}(b_i)^{-1} \\ &= \mathcal{A}_\Sigma(\phi_1, \mathcal{T}_{11}, \dots, \mathcal{T}_{k1}) \cdot \mathcal{A}_\Sigma(\phi_1, \mathcal{T}_{12}, \dots, \mathcal{T}_{k2}) \cdot \exp 2\pi i \left(\int_\Sigma -(\phi_1, \phi_2)^* \rho \right). \quad (6.1)\end{aligned}$$

The product of $f_i^* \mathcal{T}_{i1} \in \mathcal{L}_{r_i(\phi_1)}$ and $f_i^* \mathcal{T}_{i2} \in \mathcal{L}_{r_i(\phi_2)}$ is $f_i^* \mathcal{T}_i$. The claim follows by computing $s_\Sigma(\phi_1\phi_2)$ using (6.1). \square

The meaning of the calculation of Proposition 6.5 is that

$$\eta : C^\infty(\Sigma, G) \times C^\infty(\Sigma, G) \longrightarrow \text{U}(1) : (\phi_1, \phi_2) \longmapsto \exp 2\pi i \left(\int_\Sigma (\phi_1, \phi_2)^* \rho \right)$$

is a classifying 2-cocycle for the (topologically trivializable) central extension \mathcal{L}_Σ . Note that η is a coboundary if and only if \mathcal{L}_Σ has a multiplicative section. Also note that if G is abelian and η is a coboundary, then η is symmetric.

Example 6.6. Consider $G = \text{U}(1)$ and the central extension $\mathcal{L} = \mathcal{L}_P = \mathcal{L}_\mathbb{Z}(-1, 0, 1)$, i.e. the transgression of the trivial gerbe with multiplicative structure given by the Poincaré bundle P over $T = \text{U}(1) \times \text{U}(1)$, see Examples 2.5 and 4.12. By Theorem 6.3, \mathcal{L} has the smooth reciprocity property. Here, $\rho = \text{pr}_1^* \theta \wedge \text{pr}_2^* \theta \in \Omega^2(T)$. It is easy to see that we have $s^* \rho = -\rho$, where $s : T \longrightarrow T : (z, z') \longmapsto (z', z)$. This shows that η is *skew*-symmetric. But η can only be symmetric and skew-symmetric if $\eta = 1$. In order to see that this is not the case, we note that

$$\int_T \rho = 1.$$

Then, there must be an embedding $\phi : D^2 \longrightarrow T$ of a disc such that $\int \phi^* \rho \notin \mathbb{Z}$. Defining ϕ_1, ϕ_2 by composing with the projections $\text{pr}_1, \text{pr}_2 : T \longrightarrow \text{U}(1)$, we get $\eta(\phi_1, \phi_2) \neq 0$. This shows that η is not symmetric, so that \mathcal{L}_{D^2} has no multiplicative section. Hence, \mathcal{L} is a transgressive central extension that has the smooth reciprocity property but does not admit multiplicative sections.

Example 6.7. We let $G = \text{U}(1)$ and $\mathcal{L}_\mathbb{R}(\gamma)$ be the basic central extension of $LU(1)$ constructed in Example 4.12 depending on $\gamma \in \mathbb{R}$. It is clear that $\mathcal{L}_\mathbb{R}(\gamma)$ has the smooth reciprocity property, because it is topologically trivial. On the other hand, we have seen in Example 4.12 that it cannot be equipped with the structure of a thin fusion extension unless $\gamma \in \mathbb{Z}$. By Proposition 5.2.2, it is hence not transgressive. In other words, the smooth reciprocity property is not sufficient to characterize transgressive central extensions.

The conclusion of this section is that the reciprocity property (in its original form or in the version of Definition 6.1) does not properly characterize transgressive central extensions of non-complex Lie groups. The theory of loop fusion and thin homotopy equivariance that we have developed in this article provides such characterization, valid for all connected Lie groups.

A Regression of trivial fusion bundles

In this appendix we discuss the regression of trivial bundles with trivial fusion products (but non-trivial connections) over the loop space LX of a connected smooth manifold X . For this purpose, we restrict

the constructions of [Walb, Sections 5 and 6] to that case; this has not yet been worked out explicitly.

For $x \in X$ we consider the diffeological space $P_x X$ of paths in X starting at x with sitting instants, equipped with the subduction $\text{ev}_1 : P_x X \rightarrow X : \gamma \mapsto \gamma(1)$ (the diffeological analog of a surjective submersion). Two paths with the same end point compose to a loop via the smooth map $\cup_x : P_x X^{[2]} \rightarrow LX : (\gamma_1, \gamma_2) \mapsto \overline{\gamma_2} \star \gamma_1$.

Suppose $\epsilon \in \Omega^1(LX)$ is a superficial connection on the trivial bundle \mathbf{I} that is fusive with respect to the trivial fusion product. The regression $\mathcal{R}_x^\nabla(\mathbf{I}_\epsilon)$ is a bundle gerbe with connection over X , composed of the subduction $\text{ev}_1 : P_x X \rightarrow X$, the principal S^1 -bundle with connection $\cup_x^* \mathbf{I}_\epsilon$, and the identity bundle gerbe product, which is connection-preserving because ϵ is fusive. The difficult part is to specify a curving: a 2-form $B_\epsilon \in \Omega^2(P_x X)$ such that $\text{pr}_2^* B_\epsilon - \text{pr}_1^* B_\epsilon = \text{curv}(\cup_x^* \mathbf{I}_\epsilon) = \cup_x^* d\epsilon$.

Such a curving can be constructed because ϵ is superficial, see [Walb, Section 5.2]. The construction uses a bijection between the 2-forms on a diffeological space Y and certain smooth maps $\mathcal{B}Y \rightarrow \text{U}(1)$ on the space $\mathcal{B}Y$ of bigons in Y . A *bigon* is a smooth fixed-ends homotopy Σ between two paths in Y , and the correspondence between a smooth map $G : \mathcal{B}Y \rightarrow \text{U}(1)$ and a 2-form $B \in \Omega^2(Y)$ is established by the relation

$$G(\Sigma) = \exp 2\pi i \left(- \int_\Sigma B \right). \quad (\text{A.1})$$

Suppose $\Sigma \in \mathcal{B}P_x X$ is a bigon between a path $\gamma_0 \in PP_x X$ and a path $\gamma_1 \in PP_x X$. Thus, it is a smooth map $\Sigma : [0, 1]^2 \rightarrow P_x X$ such that $\Sigma(0, t) = \gamma_0(t)$ and $\Sigma(1, t) = \gamma_1(t)$. For each $t \in [0, 1]$ we

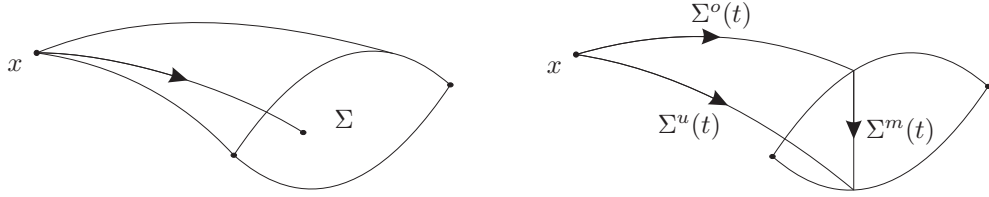


Figure 3: The picture on the left shows a bigon Σ in $P_x X$: it can be regarded as a bigon in X that has for each of its points a chosen path connecting x with that point. The picture on the right shows the three paths associated to a bigon Σ and $t \in [0, 1]$.

extract a loop $\gamma_\Sigma(t) \in LX$ defined by

$$\gamma_\Sigma(t) := (\Sigma^m(t) \star \Sigma^o(t)) \cup (\text{id} \star \Sigma^u(t)), \quad (\text{A.2})$$

where $\Sigma^m(t)$, $\Sigma^o(t)$, and $\Sigma^u(t)$ are the three paths depicted in Figure 3. Thus, γ_Σ is a path in LX that starts and ends at flat loops.

Using a smoothing function $\phi : [0, 1] \rightarrow [0, 1]$, we define a parameterized version Σ_σ for $\sigma \in [0, 1]$ by $\Sigma_\sigma(s, t) := \Sigma(\phi(s)\sigma, t)$, i.e. Σ_0 is the identity bigon at the path γ_0 , and $\Sigma_1 = \Sigma$. Now we consider $h : [0, 1]^2 \rightarrow LX : (\sigma, t) \mapsto \gamma_{\Sigma_\sigma}(t)$. By [Walb, Lemma 5.2.1] the curving B_ϵ corresponds to the smooth map

$$G_\epsilon : \mathcal{B}P_x X \rightarrow \text{U}(1) : \Sigma \mapsto \exp 2\pi i \left(- \int_{[0, 1]^2} h^* \text{curv}(\mathbf{I}_\epsilon) \right).$$

As $\text{curv}(\mathbf{I}_\epsilon) = d\epsilon$ we want to apply Stokes' theorem. Along the boundary of $[0, 1]^2$, the map h is as follows: $h(0, t) = \gamma_0(t) \cup \gamma_0(t)$, $h(1, t) = \gamma_\Sigma(t)$, $h(\sigma, 0) = \gamma_0(0) \cup \gamma_0(0)$, and $h(\sigma, 1) = \gamma_0(1) \cup \gamma_0(1)$. As

ϵ is fusive, we have $\flat^*\epsilon = 0$, where $\flat : PX \longrightarrow LX$ is the inclusion of flat loops, see Section 3.1. Thus, we get

$$G_\epsilon(\Sigma) = \exp 2\pi i \left(- \int_0^1 \gamma_\Sigma^* \epsilon \right).$$

The regressed bundle gerbe $\mathcal{R}_x^\nabla(\mathbf{I}_\epsilon)$ is not the trivial bundle gerbe (that one would have the identity subduction id_X). It is trivializable, but not canonically trivializable. However, a trivialization \mathcal{T}_κ can be obtained from a path splitting $\kappa \in \Omega^1(PX)$ of ϵ , see Definition 4.4. The trivialization \mathcal{T}_κ is composed of the principal $U(1)$ -bundle $\mathbf{I}_{-\kappa}$ over $P_x X$ and of the bundle isomorphism

$$\text{id} : \cup_x^* \mathbf{I}_\epsilon \otimes \text{pr}_2^* \mathbf{I}_{-\kappa} \longrightarrow \text{pr}_1^* \mathbf{I}_{-\kappa},$$

over $P_x X^{[2]}$, which is connection-preserving due to the defining property of a path splitting. There exists a unique 2-form $\rho_\kappa \in \Omega^2(X)$ such that $\mathcal{T}_\kappa : \mathcal{R}_x^\nabla(\mathbf{I}_\epsilon) \longrightarrow \mathcal{I}_{\rho_\kappa}$ is a connection-preserving isomorphism; this 2-form is characterized by the condition $\text{ev}_1^* \rho_\kappa = B_\epsilon - d\kappa$.

Lemma A.1. *Suppose $\rho \in \Omega^2(X)$ and $\epsilon := \tau_{S^1}(\rho) \in \Omega^1(LX)$. Then, $\kappa := \tau_{[0,1]}(\rho)$ is a path splitting for ϵ , and $B_\epsilon = \text{ev}_1^* \rho + d\kappa$. In particular, $\rho_\kappa = \rho$.*

Proof. That κ is a path splitting for ϵ has been checked in Example 4.5. Let Σ be a bigon in $P_x X$ between a path γ_0 and a path γ_1 . We have

$$G_\epsilon(\Sigma) = \exp 2\pi i \left(- \int_0^1 \gamma_\Sigma^* \epsilon \right) = \exp 2\pi i \left(\int_{[0,1] \times S^1} h_{\gamma_\Sigma}^* \rho \right)$$

with $h_{\gamma_\Sigma} : [0,1] \times S^1 \longrightarrow X$ defined by $h_{\gamma_\Sigma}(t, z) := \gamma_\Sigma(t)(z)$. We obtain from the definition (A.2) of γ_Σ :

$$h_{\gamma_\Sigma}(t, z) = \begin{cases} \gamma_0(t)(4z) & \text{if } 0 \leq z \leq \frac{1}{4} \\ \text{ev}_1(\Sigma(4z-1)(t)) & \text{if } \frac{1}{4} \leq z \leq \frac{1}{2} \\ \gamma_1(t)(3-4z) & \text{if } \frac{1}{2} \leq z \leq \frac{3}{4} \\ x & \text{if } \frac{3}{4} \leq z \leq 1 \end{cases}$$

Splitting the domain of integration into those four parts and taking care with the involved orientations yields

$$\exp 2\pi i \left(\int_{[0,1] \times S^1} h_{\gamma_\Sigma}^* \rho \right) = \exp 2\pi i \left(- \int_\Sigma \text{ev}_1^* \rho + \int_{\gamma_0} \kappa - \int_{\gamma_1} \kappa \right).$$

Finally, Stokes' theorem gives

$$\int_\Sigma d\kappa = - \int_{\gamma_0} \kappa + \int_{\gamma_1} \kappa.$$

All together, we obtain

$$G_\epsilon(\Sigma) = \exp 2\pi i \left(- \int_\Sigma (\text{ev}_1^* \rho + d\kappa) \right).$$

Using (A.1), we get the claimed equality. \square

As regression is a monoidal functor, we want to make sure that the trivialization \mathcal{T}_κ is compatible with that monoidal structure.

Lemma A.2. *Suppose ϵ_1, ϵ_2 are superficial connections on the trivial bundle \mathbf{I} over LX , and fusive with respect to the trivial fusion product. Suppose κ_1 and κ_2 are path splittings for ϵ_1 and ϵ_2 , respectively. Then, we have $\rho_{\kappa_1+\kappa_2} = \rho_{\kappa_1} + \rho_{\kappa_2}$, and there exists a connection-preserving transformation*

$$\begin{array}{ccc} \mathcal{R}_x^\nabla(\mathbf{I}_{\epsilon_1+\epsilon_2}) & \xrightarrow{\mathcal{T}_{\kappa_1+\kappa_2}} & \mathcal{I}_{\rho_{\kappa_1+\kappa_2}} \\ \downarrow \cong & \swarrow & \parallel \\ \mathcal{R}_x^\nabla(\mathbf{I}_{\epsilon_1}) \otimes \mathcal{R}_x^\nabla(\mathbf{I}_{\epsilon_2}) & \xrightarrow{\mathcal{T}_{\kappa_1} \otimes \mathcal{T}_{\kappa_2}} & \mathcal{I}_{\rho_{\kappa_1}} \otimes \mathcal{I}_{\rho_{\kappa_2}} \end{array}$$

Proof. The isomorphism $\mathcal{R}_x^\nabla(\mathbf{I}_{\epsilon_1+\epsilon_2}) \cong \mathcal{R}_x^\nabla(\mathbf{I}_{\epsilon_1}) \otimes \mathcal{R}_x^\nabla(\mathbf{I}_{\epsilon_2})$ that implements that \mathcal{R}_x^∇ is monoidal is induced from the connection-preserving, fusion-preserving isomorphism $\text{id} : \mathbf{I}_{\epsilon_1+\epsilon_2} \rightarrow \mathbf{I}_{\epsilon_1} \otimes \mathbf{I}_{\epsilon_2}$. In particular, we have $B_{\epsilon_1+\epsilon_2} = B_{\epsilon_1} + B_{\epsilon_2}$. We calculate $\text{ev}_1^*(\rho_{\kappa_1} + \rho_{\kappa_2}) = B_{\epsilon_1} - d\kappa_1 + B_{\epsilon_1} - d\kappa_1 = B_{\epsilon_1+\epsilon_2} + d(\kappa_1 + \kappa_2)$; this shows that $\rho_{\kappa_1+\kappa_2} = \rho_{\kappa_1} + \rho_{\kappa_2}$. The announced connection-preserving transformation is now simply induced by the connection-preserving isomorphism $\text{id} : \mathbf{I}_{-(\kappa_1+\kappa_2)} \rightarrow \mathbf{I}_{-\kappa_1} \otimes \mathbf{I}_{-\kappa_2}$. \square

The next two propositions describe the relation between the trivialization \mathcal{T}_κ of the regressed bundle gerbe $\mathcal{R}_x^\nabla(\mathbf{I}_\epsilon)$, the canonical trivialization t_ρ of $\mathcal{T}_{\mathcal{I}_\rho}^\nabla$, and the two natural equivalences

$$\mathcal{A} : \mathcal{R}_x^\nabla \circ \mathcal{T}^\nabla \rightarrow \text{id}_{\text{h}_1\text{Grb}^\nabla(X)} \quad \text{and} \quad \varphi : \mathcal{T}^\nabla \circ \mathcal{R}_x^\nabla \rightarrow \text{id}_{\mathcal{FusBun}^\nabla\text{sf}(X)}$$

that establish that the functors \mathcal{R}_x^∇ and \mathcal{T}^∇ form an equivalence of categories [Walb, Theorem A].

Proposition A.3. *Suppose $\rho \in \Omega^2(X)$. Let $t_\rho : \mathcal{T}_{\mathcal{I}_\rho}^\nabla \rightarrow \mathbf{I}_\epsilon$ be the canonical trivialization, with $\epsilon = \tau_{S^1}(\rho) \in \Omega^1(LX)$. Let $\mathcal{A}_{\mathcal{I}_\rho} : \mathcal{R}_x^\nabla(\mathcal{T}_{\mathcal{I}_\rho}^\nabla) \rightarrow \mathcal{I}_\rho$ be the component of the natural equivalence \mathcal{A} at \mathcal{I}_ρ . Let $\kappa := \tau_{[0,1]}(\rho)$ be the canonical path splitting of ϵ and let $\mathcal{T}_\kappa : \mathcal{R}_x^\nabla(\mathbf{I}_\epsilon) \rightarrow \mathcal{I}_\rho$ be the corresponding trivialization. Then, there exists a connection-preserving transformation*

$$\mathcal{A}_{\mathcal{I}_\rho} \cong \mathcal{T}_\kappa \circ \mathcal{R}_x^\nabla(t_\rho).$$

Proof. The isomorphism $\mathcal{R}_x^\nabla(t_\rho) : \mathcal{R}_x^\nabla(\mathcal{T}_{\mathcal{I}_\rho}^\nabla) \rightarrow \mathcal{R}_x^\nabla(\mathbf{I}_\epsilon)$ is induced from the bundle isomorphism $\cup^* t_\rho : \cup^* \mathcal{T}_{\mathcal{I}_\rho}^\nabla \rightarrow \mathbf{I}_{\cup^* \epsilon}$ over $P_x X^{[2]}$. The composition $\mathcal{T}_\kappa \circ \mathcal{R}_x^\nabla(t_\rho)$ is thus given by the S^1 -bundle $\mathbf{I}_{-\kappa}$ and the isomorphism

$$\cup_x^* \mathcal{T}_{\mathcal{I}_\rho}^\nabla \otimes \text{pr}_2^* \mathbf{I}_{-\kappa} \xrightarrow{\cup_x^* t_\rho \otimes \text{id}} \cup_x^* \mathbf{I}_\epsilon \otimes \text{pr}_2^* \mathbf{I}_{-\kappa} \xrightarrow{\text{id}} \text{pr}_1^* \mathbf{I}_{-\kappa}. \quad (\text{A.3})$$

Next we describe the connection-preserving isomorphism $\mathcal{A}_{\mathcal{I}_\rho}$ following [Walb, Section 6.1]. It consists of an S^1 -bundle Q over $P_x X$ with connection, and of a connection-preserving bundle isomorphism $\alpha : \cup_x^* \mathcal{T}_{\mathcal{I}_\rho}^\nabla \otimes \text{pr}_2^* Q \rightarrow \text{pr}_1^* Q$ over $P_x X^{[2]}$.

The fibre of Q over $\gamma \in P_x X$ consists of triples (\mathcal{T}, t_0, t) , where $\mathcal{T} : \gamma^* \mathcal{I}_\rho \rightarrow \mathcal{I}_0$ is a trivialization (in turn consisting of an S^1 -bundle T with connection τ over $[0, 1]$ and of a connection-preserving bundle isomorphism which here is necessarily the identity $\tau = \text{id}_T$), $t_0 \in T_0$ and $t \in T_1$. Two triples (\mathcal{T}, t_0, t) and (\mathcal{T}', t'_0, t') are identified if there exists a connection-preserving transformation $\varphi : \mathcal{T} \Rightarrow \mathcal{T}'$ such

that $\varphi(t_0) = t'_0$ and $\varphi(t) = t'$. The S^1 -action on S^1 is $(\mathcal{T}, t_0, t) \cdot z := (\mathcal{T}, t_0, t \cdot z)$. In our situation, Q has a canonical section $s : P_x X \rightarrow Q : (\gamma, z) \mapsto (\text{id}_{\mathcal{I}_0}, 1, 1)$, using that $\gamma^* \mathcal{I}_\rho = \mathcal{I}_{\gamma^* \rho} = \mathcal{I}_0$. The bundle isomorphism α is over a point $(\gamma_1, \gamma_2) \in P_x X^{[2]}$ a map

$$\alpha : \mathcal{T}_{\mathcal{I}_\rho}|_{\gamma_1 \cup \gamma_2} \otimes Q_{\gamma_2} \rightarrow Q_{\gamma_1},$$

and it is characterized by $\alpha(t_\rho(\gamma_1 \cup \gamma_2) \otimes s(\gamma_2)) = s(\gamma_1)$. The connection on Q is defined via its parallel transport. Using the section $s : P_x X \rightarrow Q$, it suffices to define a 1-form on $P_x X$, and we do this by defining a smooth map $F : PP_x X \rightarrow S^1$. This map is given by

$$F(\gamma) = \int_{h_\gamma} \rho,$$

where $h_\gamma : [0, 1]^2 \rightarrow X$ is defined by $h_\gamma(s, t) = \gamma(s)(t)$. However, this map characterizes precisely the parallel transport of the 1-form $-\kappa = -\tau_{[0, 1]}(\rho) \in \Omega^1(P_x X)$.

Summarizing, s defines a connection-preserving transformation between $\mathcal{A}_{\mathcal{I}_0}$ and the trivialization consisting of the trivial bundle $\mathbf{I}_{-\kappa}$ and of the isomorphism (A.3). \square

Proposition A.4. *Suppose $\epsilon \in \Omega^1(LX)$ is a superficial connection on the trivial $U(1)$ -bundle over LX , and fusive with respect to the trivial fusion product. Let $\varphi_{\mathbf{I}_\epsilon} : \mathcal{T}_{\mathcal{R}_x^\nabla(\mathbf{I}_\epsilon)}^\nabla \rightarrow \mathbf{I}_\epsilon$ be the component of the natural equivalence φ at \mathbf{I}_ϵ . Let $\kappa \in \Omega^1(PX)$ be a contractible path splitting for ϵ , and let $\mathcal{T}_\kappa : \mathcal{R}_x^\nabla(\mathbf{I}_\epsilon) \rightarrow \mathcal{I}_{\rho_\kappa}$ be the corresponding trivialization. Let $t_{\rho_\kappa} : \mathcal{T}_{\mathcal{I}_{\rho_\kappa}}^\nabla \rightarrow \mathbf{I}_{\epsilon_\kappa}$ be the canonical trivialization with $\epsilon_\kappa = \tau_{S^1}(\rho_\kappa)$. Then,*

$$\varphi_{\mathbf{I}_\epsilon} = t_{\rho_\kappa} \circ \mathcal{T}_{\mathcal{T}_\kappa}^\nabla; \quad (\text{A.4})$$

in particular, $\epsilon_\kappa = \epsilon$.

Proof. Note that (A.4) is an equality between two connection-preserving bundle isomorphisms going from $\mathcal{T}_{\mathcal{R}_x^\nabla(\mathbf{I}_\epsilon)}^\nabla$ to \mathbf{I}_ϵ and $\mathbf{I}_{\epsilon_\kappa}$, respectively. This implies $\epsilon = \epsilon_\kappa$. For $\beta \in LG$ a loop, $\beta^* \mathcal{T}_\kappa$ is a trivialization of $\beta^* \mathcal{R}_x^\nabla(\mathbf{I}_\epsilon)$, and thus an element of $\mathcal{T}_{\mathcal{R}_x^\nabla(\mathbf{I}_\epsilon)}^\nabla$ over β . We have

$$\mathcal{T}_{\mathcal{T}_\kappa}^\nabla(\beta^* \mathcal{T}_\kappa) = \beta^* \mathcal{T}_\kappa \circ \beta^* \mathcal{T}_\kappa^{-1} = \text{id}_{\mathcal{I}_{\beta^* \rho_\kappa}},$$

considered as an element of $\mathcal{T}_{\mathcal{I}_{\rho_\kappa}}^\nabla$. Under the canonical trivialization t_{ρ_κ} , this element is equal to $(\beta, 1) \in LX \times U(1)$.

On the other hand, we compute the element $p := \varphi_{\mathbf{I}_\epsilon}(\beta^* \mathcal{T}_\kappa) \in \mathbf{I}_\epsilon$ following the definition of φ given in [Walb, Section 6.2]. We have to consider the space $Z := S^1 \times_{\beta \times \text{ev}_1} P_x X$ as the subduction of $\beta^* \mathcal{R}_x^\nabla(\mathbf{I}_\epsilon)$ and over Z the bundle $\mathbf{I}_{-\kappa}$, pulled back along the projection $Z \rightarrow P_x X$. Over $Z \times_{S^1} Z$ the trivialization $\beta^* \mathcal{T}_\kappa$ has the identity morphism

$$\text{id} : \cup_x^* \mathbf{I}_\epsilon \otimes \text{pr}_2^* \mathbf{I}_{-\kappa} \rightarrow \text{pr}_1^* \mathbf{I}_{-\kappa},$$

also pulled back along $Z \times_{S^1} Z \rightarrow P_x X^{[2]}$. We represent the loop β by a path $\gamma \in P_x X$ with $\gamma(1) = \beta(0)$ and paths $\gamma_k \in PX$ with $\gamma_k(0) = \beta(0)$ and $\gamma_k(1) = \beta(\frac{1}{2})$, related via a thin homotopy $h : (\gamma_1 \star \gamma) \cup (\gamma_2 \star \gamma) \rightarrow \beta$. In Z we consider the retracting paths α_i with $\alpha_i(0) = (0, \text{id} \star \gamma)$ and $\alpha_i(1) = (\frac{1}{2}, \gamma_i \star \gamma)$. Then, the prescription is

$$p = (\beta, \exp 2\pi i \left(- \int_{\alpha_2 \star \alpha_1} \text{pr}^* \kappa \right)) = (\beta, \exp 2\pi i \left(\int_{\alpha_1} \text{pr}^* \kappa - \int_{\alpha_2} \text{pr}^* \kappa \right)).$$

Since the paths α_i are retractions and the path splitting κ is contractible, both integrals vanish separately. Thus, we have $p = (\beta, 1)$; this yields the claimed equality. \square

Table of terminology

Fusion product	$\lambda : \mathcal{L}_{\gamma_1 \cup \gamma_2} \otimes \mathcal{L}_{\gamma_2 \cup \gamma_3} \longrightarrow \mathcal{L}_{\gamma_1 \cup \gamma_3}$	Page 4
— multiplicative	λ is a group homomorphism	Page 4
Thin homotopy equivariant structure		
	$d_{\tau_0, \tau_1} : \mathcal{L}_{\tau_0} \longrightarrow \mathcal{L}_{\tau_1}$ for thin homotopic loops τ_0, τ_1	Page 5
— multiplicative	d is a group homomorphism	Page 5
— compatible	d is connection-preserving	Page 5
— symmetrizing	rotation by an angle of π switches factors under fusion	Page 5
— fusive	compatible and symmetrizing	Page 5
Bundle morphism	$\varphi : \mathcal{L} \longrightarrow \mathcal{L}'$	
— fusion-preserving	$\varphi(\lambda(p \otimes q)) = \lambda'(\varphi(p) \otimes \varphi(q))$	Page 4
— thin	$\varphi(d_{\tau_0, \tau_1}(p)) = d'_{\tau_0, \tau_1}(\varphi(p))$	Page 5
Connection		
— thin	induces a thin homotopy equivariant structure	Page 13
— superficial	thin and its holonomy is thin homotopy invariant	Page 13
— compatible	fusion product is connection-preserving	Page 14
— symmetrizing	induced thin homotopy equivariant structure is symmetrizing	Page 14
— fusive	compatible and symmetrizing	Page 14
Path splitting of $\epsilon \in \Omega^k(LG \times LG)$		
	$\kappa \in \Omega^k(PG \times PG)$ with $\epsilon_{\gamma_1 \cup \gamma_2, \gamma'_1 \cup \gamma'_2} = \kappa_{\gamma_1, \gamma'_1} - \kappa_{\gamma_2, \gamma'_2}$	Page 14
— multiplicative	$\kappa_{\gamma_1 \gamma'_1, \gamma_2 \gamma'_2} = \kappa_{\gamma_1, \gamma_2} + \kappa_{\gamma'_1, \gamma'_2}$	Page 14
— contractible	$\int_{\phi_\gamma} \kappa = 0$, where ϕ_γ is the contraction of a path γ	Page 14
Thin structure	thin homotopy equivariant structure induced by a superficial connection	Page 14
— fusive	an inducing connection is fusive	Page 14
— multiplicative	the error 1-form of an inducing connection admits a multiplicative and contractible path splitting	
— fusive and multiplicative	there is an inducing connection with both of above properties	Page 15
Central extension of LG		
— thin fusion	equipped with a multiplicative fusion product and a fusive and multiplicative thin structure	Page 15

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